Logic and homotopy in the category of assemblies

The final report for an individual project under the supervision of prof Jaap van Oosten[1]

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March 18, 2016

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Chapter 1

Introduction

The notion of *realizability* first appeared in Kleene's [6]. Original Kleene's realizability interpretation (now known as number realizability, provided a formalized version of the BHK interpretation, a way of extracting algorithmic content of constructive proofs. For instance, from a proof of the statement $\exists x.Q(x)$ one would want to extract a witness for the existential quantifier, a number n, s.t. $Q(n)$ holds; or from a proof of a statement $\forall x.\exists y.P(x,y)$ one would want to extract a computable function f, such that $P(m, f(m))$ for all numbers m.

The Kleene realizability relation $\underline{\mathsf{nr}}$ links together numbers and formulas of first-order intuitionistic number theory, and is defined by induction on the formula.

- $n \ \underline{\mathsf{nr}}\ (p = q)$ if $\vdash p = q$;
- $n \ \underline{\mathsf{nr}}\ (A \wedge B)$ if $\mathsf{p}_1 n \ \underline{\mathsf{nr}}\ A$ and $\mathsf{p}_2 n \ \underline{\mathsf{nr}}\ B$;
- $n \ \underline{\mathsf{nr}}\ (A \vee B)$ if $\mathsf{p}_1 n = 0$ and $\mathsf{p}_2 n \ \underline{\mathsf{nr}}\ A$ or $\mathsf{p}_1 n = 1$ and $\mathsf{p}_2 n \ \underline{\mathsf{nr}}\ B$;
- $n \ \underline{\mathsf{nr}}\ (A \Rightarrow B)$ if for all m, s.t. $m \ \underline{\mathsf{nr}}\ A$, partial recursive function $\{n\}$ is defined at m and $\{n\}m \ \underline{\mathsf{nr}}\ B$;
- $n \ \underline{\mathsf{nr}}\ \forall x.A(x)$ if for all $m \in \mathbf{N}$, partial recursive function $\{n\}$ is defined at m and $\{n\}m \ \underline{\mathsf{nr}}\ A(\bar{m})$;
- $n \ \underline{\mathsf{nr}}\ \exists x.A(x)$ if $\mathsf{p}_2 n \ \underline{\mathsf{nr}}\ A(\overline{\mathsf{p}_1 n})$.

where $\{-\}-$ denotes Kleene application, p_1 and p_2 denote primitive recursive (p.r.) projection functions , and $\bar{m}$ is the representation of number m inside the formal system.

One can prove, by induction on proofs in **HA**, that the number realizability interpretation is *sound*, that is, if a (closed) formula φ is derivable in **HA**, then there is a number n, s.t. $n \ \underline{\mathsf{nr}}\ \varphi$. Viewing number realizability as a formalization of the BHK principle, we can say that the realizability relation gives us a notion of *formal semantics* for intuitionistic number theory; however, the resulting logic is different from the logic of **HA**. For instance, the Church thesis

$$\forall x.\exists y.P(x, y) \Rightarrow \exists e.\forall x.P(x, \{e\}x) \tag{1.1}$$

is realizable (the realizer for the antecedent gives the witness for $\exists e$); however, it is not provable in **HA**. For instance, take $P(x, y)$ to say that $y = 0 \Rightarrow$ the x-th Turing machine terminates on the input x and $y \neq 0 \Rightarrow$ that the x-th Turing machine does not terminate on the input x. If CT was provable in **HA**, it would also be provable in **PA**. We know, that classically, each Turing machine either terminates or not, however, the p.r. function $\{e\}$ would give us a way of deciding whether a given machine x terminates on x or not – which is contradictory with elementary recursion theory (which holds in **HA**). The reader who wishes to learn more about Kleene's number realizability is referred to [12],[13, Volume 2], and [10].

The study of realizability from a categorical perspective, arguably, began with the publication [5]. The so-called effective topos can be viewed as an intuitionistic or constructive universe. In this document we mainly focus on a specific subcategory of the effective topos – the category of assemblies, which is suitable for first-order reasoning.

The document is structured as follows. The next chapter is devoted to categorical logic inside the category of assemblies. First, we discuss notion of a *hyperdoctrine* – a way of interpreting logic in a category, which was first introduced by Lawvere in his seminal work [3]. Then, in section 2.2 we construct the category of *assemblies* and realizable maps between them. After that, in section 2.3 we describe the hyperdoctrine structure on the category of assemblies, roughly following the notes [11]. We prove the completeness theorem, saying that a formula of arithmetic is valid in the category of assemblies iff it is realizable in the sense of Kleene. The author did not have any novel contributions to the material recalled in the first chapter.

The second (and the last) chapter deals with a notion of homotopy in the category of assemblies and a view of assemblies as topological spaces. We build on the notion of homotopy in the effective topos [7]. The main differences between the presentation in [7] and in this text are

1. We restrict ourselves to the category of assemblies;
2. We make explicit and wider use of the unifying theory of path object categories [1]. In addition, we generalize some statements to arbitrary path object categories.

In section 3.1 we recall the definition of a (nice) path object category and show that the category of assemblies posses the required structure, ensuring that the results developed later in the text apply. Then, in section 3.2 we describe a notion of homotopy and construct the "extended" path contraction natural transformation, working in an arbitrary path object category. In section 3.3 we give the definition of a Hurewicz fibration in terms of a Hurewicz connection, and characterize Hurewicz fibrations by the homotopy lifting property. We finish the section by presenting a (strong deformation retract, Hurewicz fibration) functorial factorisation. Finally, the last section is devoted to the description of discrete objects, which are viewed as discrete spaces.

Before we begin, let us recall standard recursion-theoretic notation. Given natural numbers n, m we write $\{n\}(m)$ or $n \cdot m$ for Kleene application. As usual, application is left-associative. We use the notation $\langle -, - \rangle$ for primitive recursive pairing, and p_1, p_2 for primitive recursive projections. We write recursively encoded sequences like $\langle a, b, c \rangle$. Usually we reserve Greek letters for numbers representing recursive functions, and we make liberal use of λ-notation.

We use $\mathrm{rec}(-;-;-)$ for primitive recursion, i.e. $\mathrm{rec}(0;a;b) = 0$ and $\mathrm{rec}(x+1;a;b) = b \cdot x \cdot (\mathrm{rec}(x;a;b))$.

Chapter 2

Categorical realizability logic

In this chapter we discuss the categorical logic arising in the category of *assemblies*. First, we recall the theory of *hyperdoctrines* as developed by Lawvere [3, 4] and Seely [9]. Then we describe the category *Asm* of assemblies, as presented in [11] and [8], including explicit formulation of certain limits and colimits. Finally we reconstruct in detail the interpretation of first-order logic and *Asm* and showing that it is sound and complete with regard to Kleene's number realizability interpretation.

2.1 Hyperdoctrines

Consider a *typed/many-sorted* first-order theory. The judgments in the theory are of the form

$$x : A, y : B, \dots, z : C \vdash t(x, y, \dots, z) : D$$

meaning that under a context $x : A, y : B, \dots, z : C$ the term t has a type D, and

$$x : A, y : B, \dots z : C \vdash \varphi(x, y, \dots, z) \leq \psi(x, y, \dots, z)$$

meaning that under a context $x : A, y : B, \dots, z : C$ the formula φ entails the formula ψ.

We use a shorthand $\Gamma \vdash \phi$ to mean $\Gamma \vdash \top \leq \phi$.

To motive a definition of a hyperdoctrine, let us first consider set-theoretic interpretation of first-order logic, that is interpretation in the category **Set** of sets. If we view sorts as sets, then a conext $x_1 : A_1, \dots, x_n : A_n$ determines a set $A_1 \times \cdots \times A_n$. Then, a specific formula $\varphi(x_1, \dots, x_n)$ determines a subset $[\![\varphi]\!]$ of $A_1 \times \cdots \times A_n$, namely the subset of those elements that satisfy φ. Conversely, we can think of each subset as a generalized formula. To put it another way, a formula φ is interpreted as an element of a Boolean algebra $\mathcal{P}(A_1 \times \cdots \times A_n)$. Given two formulas φ and ψ over the same context, we can form their conjunction $\varphi \wedge \psi$, which corresponds to the meet operation on the powerset algebra (intersection of subsets). Similarly, other Boolean algebra operations correspond to disjunction, implication, negation. We illustrate the correspondence in a table below.

$$\begin{array}{rcl} \varphi \wedge \psi & \sim & [\![\varphi]\!] \cap [\![\psi]\!] \\ \varphi \vee \psi & \sim & [\![\varphi]\!] \cup [\![\psi]\!] \\ \neg\varphi & \sim & (A_1 \times \cdots \times A_n) \setminus [\![\varphi]\!] \end{array}$$

The next aspect that we want to consider are terms. Categorically, a term $t(x_1, \dots, x_n)$ of type B under a context $x_1 : A_1, \dots, x_n : A_n$ is interpreted as a morphism $[\![t]\!] : A_1 \times \cdots \times A_n \to B$, where the object $A_1 \times \cdots \times A_n$ corresponds to the context $x_1 : A_1, \dots, x_n : A_n$.

Once we have settled with terms, we can extend the categorical interpretation to substitution. A substitution $\delta : x_i \mapsto t_i(y_1, \dots, y_m)$ of variables $x_1, \dots, x_n$, with $x_i : A_i$ and $y_i : B_i$ can be seen as "generalized term" in the context $y_1 : B_1, \dots, y_m : B_m$, or a morphism $B_1 \times \cdots \times B_m \xrightarrow{(t_1, \dots, t_n)} A_1 \times \cdots \times A_n$. Given a term $s(x_1, \dots, x_n)$ of sorts C with free variables of sorts $A_1, \dots, A_n$, we can substitute for each x_i a term t_i, resulting in a term

$$s(x_1, \dots, x_m)[\delta] = s(t_1, \dots, t_n)$$

in the context $B_1, \dots, B_m$, which is defined as a composition

$$B_1 \times \cdots \times B_m \xrightarrow{\delta} A_1 \times \cdots \times A_n \xrightarrow{s} C$$

However, the substitution of terms for variables in a formula is defined differently. Suppose that $\phi(x_1, \dots, x_n)$ is a formula in the context $A_1 \times \cdots \times A_n$. Then the substituted formula $\phi(x_1, \dots, x_n)[\delta] = \phi(t_1, \dots, t_n)$ is defined as a pullback of the inclusion $[\![\phi]\!] \hookrightarrow A_1 \times \cdots \times A_n$ along the substitution map $\delta : B_1 \times \cdots \times B_m \to A_1 \times \cdots \times A_n$.

$$\begin{tikzcd}
\llbracket \phi[\delta] \rrbracket \arrow[rr] \arrow[d, hook] & & \llbracket \phi \rrbracket \arrow[d, hook] \\
B_1 \times \cdots \times B_m \arrow[rr, "\delta"] & & A_1 \times \cdots \times A_n
\end{tikzcd}$$

An explicit calculation in **Set** would give us $\llbracket \phi[\delta] \rrbracket = \delta^*(\llbracket \phi \rrbracket)$ as a subset of $(B_1 \times \cdots \times B_m) \times \llbracket \phi \rrbracket$, specifically

$$\llbracket \phi[\delta] \rrbracket = \{(y_1, \ldots, y_m, x_1, \ldots, x_n) \in (B_1 \times \cdots \times B_m) \times \llbracket \phi \rrbracket \mid \delta(y_1, \ldots, y_m) = (x_1, \ldots, x_n)\}$$

Which is isomorphic to the set

$$\begin{aligned}
&\{(y_1, \ldots, y_m) \in B_1 \times \cdots \times B_m \mid \delta(y_1, \ldots, y_m) \in \llbracket \phi \rrbracket\} \\
&= \{(y_1, \ldots, y_m) \in B_1 \times \cdots \times B_m \mid (t_1, \ldots, t_n) \in \llbracket \phi \rrbracket\}
\end{aligned}$$

because each x_i is determined by δ and $(y_1, \ldots, y_m)$.

A particular case of substitution is *weakening*, corresponding to a proof rule

$$\frac{\Gamma \vdash \phi}{\Gamma, x : X \vdash \phi}$$

saying that every formula in a context Γ can be interpreted as a formula in a context $(\Gamma, x : X)$. The substitution that corresponds to this is

$$\llbracket \Gamma \rrbracket \times X \xrightarrow{\pi_1} \llbracket \Gamma \rrbracket$$

Quantification. The proof rules for the existential and universal quantification are shown below.

$$\frac{\Gamma \vdash \exists x : X . \varphi(x, y) \leq \psi(y)}{\Gamma, x : X \vdash \varphi(x, y) \leq \psi(y)[\pi_1]}$$

$$\frac{\Gamma \vdash \varphi(y) \leq \forall x . \psi(x, y)}{\Gamma, x : X \vdash \varphi(y)[\pi_1] \leq \psi(x, y)}$$

This suggest that we interpret existential quantification as the left adjoint to π_1^*, and the universal quantification as the right adjoint to π_2^*. Generally, we require such adjoints for *any* morphism/substitution.

$$\exists_f \dashv f^* = [f] \dashv \forall_f$$

By $\exists x : X$ or $\exists_{x:X}$ we mean $\exists_{\pi_1}$, where $\pi_1 : X \times A \to A$.

This brings us to a general definition of a hyperdoctrine[1]

Definition 1 (Hyperdoctrine [4])**.** *A* hyperdoctrine *is a category $\mathcal{C}$, that has finite products, and for which the subobject construction yields a functor $Sub : \mathcal{C}^{op} \to Heyt$ into the category of Heyting algebras. Furthermore, we require that each $Sub(f)$, viewed as a map of posets, has a left and a right adjoint:*

$$\exists_f \dashv Sub(f) \dashv \forall_f$$

Remark 2. *We would like to make the following remarks.*

1. *We use the category $\mathcal{C}$ for interpreting both contexts and sorts; suppose we have a list $A_1, \ldots, A_n$ of sorts. Each A_i we interpret as $[\![A_I]\!]$ an object of $\mathcal{C}$, then the contexts are built iteratively from types, using products in $\mathcal{C}$. Sometimes it is meaningful to combine sorts (to build, e.g. sum-types, product-types, function-types), and for that we also use categorical/universal constructions (co-products, products, exponentiation) in $\mathcal{C}$.*

2. *We stress again the the functions $\exists_f$ and $\forall_f$ are not required to be Heyting algebra morphisms; merely maps of posets.*

2.1.1 Equality

The equality relation is actually definable in an arbitrary hyperdoctrine.

Let A be an object of the category of contexts, and let $\delta = \langle \mathrm{id}_A, \mathrm{id}_A \rangle : A \to A \times A$ be a *diagonal morphism*, obtained as the universal arrow arising from the following diagram:

[1] The definition below is actually slightly limited, and know in the literature as a *first-order subobject hyperdoctrine*, *Heyting category*[8] or a *logos*[9].

$$\begin{tikzcd}
A & A \times A \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & A \\
& A \arrow[ul, "\mathrm{id}_A"] \arrow[u, dotted, "\delta_A"] \arrow[ur, "\mathrm{id}_A"'] &
\end{tikzcd}$$

Then $\exists_\delta(\top_A) : Sub(A \times A)$, a relation on $A \times A$. We also denote $\exists_\delta(\top_A)$ by Eq_A or $\mathrm{Eq}_A(x, y)$. From the adjunction property we get the proof rule describing the equality relation:

$$\frac{A \times A \vdash \mathrm{Eq}_A \leq R}{A \vdash \top_A \leq R[\delta_A]}$$

i.e.

$$\frac{x : A, y : A \vdash \mathrm{Eq}_A(x, y) \leq R(x, y)}{x : A \vdash R(x, x)}$$

Thus every reflexive relation is implied by equality, and equality implies only reflexive relations.

Properties of the equality

Reflexivity. From the unit $\eta : 1_A \Rightarrow \delta_A^* \circ \exists_{\delta_A}$ we obtain a canonical derivation $\top_A \leq \delta_A^*(\mathrm{Eq}_A)$, i.e. a derivation $x : A \vdash \mathrm{Eq}_A(x, x)$; thus the equality relation is reflexive.

Symmetry. Let $s_{A,B} = \langle \pi_2, \pi_1 \rangle : A \times B \to B \times A$ be a "swapping" morphism. Note that by the universal property of the product, $s_{A,A} \circ \delta_A = \delta_A$. Then

$$\text{Functoriality of } (-)^* \; \frac{\dfrac{\dfrac{x : A \vdash \top_A \leq \delta^*(\mathrm{Eq}_A) = (s \circ \delta)^*(\mathrm{Eq}_A)}{x : A \vdash \top_A \leq \delta^*(s^*(\mathrm{Eq}_A))}}{x : A, y : A \vdash \exists_\delta \top_A \leq s^*(\mathrm{Eq}_A)}}{x : A, y : A \vdash \mathrm{Eq}_A(x, y) \leq \mathrm{Eq}_A(x, y)[s] = \mathrm{Eq}_A(y, x)}$$

Transitivity.

$$\frac{\dfrac{x : A, y : A, z : A \vdash \mathrm{Eq}_A(x, y) \wedge \mathrm{Eq}_A(y, z) \leq \mathrm{Eq}_A(x, z)}{x : A, y : A, z : A \vdash \mathrm{Eq}_A(x, y) \leq \mathrm{Eq}_A(y, z) \to \mathrm{Eq}_A(x, z)}}{x : A, y : A \vdash \top_A \leq \mathrm{Eq}_A(x, z) \to \mathrm{Eq}_A(x, z)}$$

the last line being true in in arbitrary Heyting algebra.

2.2 Category of assemblies

Definition 3. *An* assembly *is a pair* (X, E) *where* X *is an underlying* carrier set*, and* E *is the* realizability relation*, a function* $X \to \mathcal{P}(\mathbf{N})$*, s.t.* $E(x)$ *is non-empty for every* x*.*

Intuitively, if $x \in X$, then $E(X)$ is a set of reasons for the "existence" of x. Conventionally, we write E_X for the realizability relation associated with an assembly X. We also write $a \Vdash_X x$ for $a \in E_X(x)$. For an assembly X we write $|X|$ for its underlying set.

A *morphism* of assemblies (X, E_X) and (Y, Y_E) is a function $f : X \to Y$, such that there is a p.r. function $\{n\}$ that *tracks (or realizes)* f:

$$a \in E_X(x) \implies \{n\}(a) \in E_Y(f(x))$$

The assemblies form a category *Asm* of assemblies, with the identity morphism being tracked by the identity p.r. function, and the composition of morphism being tracked by composition of p.r. functions. Note that the morphism contains only the fact of the existence of the tracker/realizer, but not the tracker/realizer itself. In particular, a morphism is defined fully by its underlying function of sets.

2.2.1 Limits and colimits in *Asm*

In this subsection, we describe some common limit and colimit constructions in *Asm*.

Initial object. The initial object 0 is an assembly with the empty underlying set and an empty realizability relation. For each assembly X, the initial map $\perp : 0 \to X$ is an empty function, tracked by any number.

Products.

$$X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y$$

A product of the assemblies X and Y is an object $X \times Y$ with the underlying set $|X| \times |Y|$ and the realizability relation

$$a \Vdash_{X\times Y} (x, y) \iff \mathsf{p}_1 a \Vdash_X x \wedge \mathsf{p}_2 a \Vdash_Y y$$

The projections π_1 and π_2 are tracked by p_1 and p_2.

Pullbacks. A pullback $A \times_B C$ can be described in terms of a sub-assembly of the product.

$$\begin{tikzcd} A \times_B C \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C \arrow[d, "g"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

The underlying set of $A \times_B C$ is $\{(a, c) \in A \times C \mid f(a) = g(a)\}$; the realizability relation is given by $e \Vdash_{A\times_B C} (a, c) \iff e \Vdash_{A\times C} (a, c)$.

Equalizers.

$$\begin{tikzcd} E \arrow[r, "e"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

$|E| = \{x \in X \mid f(x) = g(x)\}$ and e is tracked by the identity p.r. function.

Exponentiation in Asm. If X and Y are assemblies, then Y^X is an assembly with a carrier set $Hom_{Asm}(X, Y)$, with the realizability relation

$$a \Vdash_{Y^X} f \iff a \text{ tracks } f$$

The evaluation map $ev : Y^X \times X \to Y$ is tracked by $\lambda x.(\mathsf{p}_1 x) \cdot (\mathsf{p}_2 x)$. To see that ev is universal, suppose that $u : Z \times X \to Y$ is a morphism of assemblies, tracked by $\underline{u}$. Then there is a unique map r making the diagram below commute.

$$\begin{tikzcd} Z \times X \arrow[r, "r\times \mathrm{id}_Y"] \arrow[dr, "u"'] & Y^X \times X \arrow[d, "ev"] \\ & Y \end{tikzcd}$$

The map r is defined as

$$r(z) = \lambda x.u(z, x)$$

and is tracked by

$$\lambda a.\lambda b.\underline{u} \cdot \langle a, b\rangle$$

2.2.2 Alternative presentation

Alternatively, we can drop the non-emptiness/"existence" condition on assemblies, and obtain the following definition.

Definition 4. *An* assembly* *is a pair (X, E) where X is an underlying set, and $E : X \to \mathcal{P}(\mathbf{N})$ is a realizability relation, such that $E(x)$ is allowed to be empty.*

A morphism between two assemblies* $(X, E_X) \to (Y, E_Y)$ is a function $\{x \in X \mid E_X(x) \neq \emptyset\} \to Y$ that is tracked, in the sense of definition 3. This definition gives rise to the category Asm^*, which is equivalent to Asm.

We will use this presentation later in the text, when talking about subobjects in Asm (Asm^*).

2.3 Logic in Asm

2.3.1 Subobjects in Asm

Definition 5. *Let X be an object of Asm. We say that a subobject $M \hookrightarrow X$ is in* canonical form *if*

1. $M \subseteq X$
2. $\langle a, b\rangle \Vdash_M x \iff a \in M_x$ *and* $b \Vdash_X x$ *for some set* $M_x \subseteq \mathbf{N}$

The inclusion $M \hookrightarrow X$ is tracked by p_2.

The terminology in the last definition is warranted by the following theorem:

Theorem 6. *Every subobject is isomorphic to a subobject in canonical form.*

Proof. Let $n : N \hookrightarrow X$ be a subobject of X, tracked by a term $\overline{n}$. We define a subobject M as a carrier set $\{x \in X \mid \exists y.n(y) = x\}$ with the realizability relation

$$\langle a, b\rangle \Vdash_M x \iff b \Vdash_X x \wedge \exists y \in N(a \Vdash_N y \wedge n(y) = x)$$

Then $f : N \to M$ is given by $f(y) = n(y)$ and is tracked by $\lambda a.\langle a, \overline{n}a\rangle$. The inverse $g : M \to N$ is defined by $g(x) = n^{-1}(x)$ and is tracked by p_1. $\square$

This proposition will allow us to work only with subobjects in canonical form, which will make our life easier in some cases.

First of all, we describe the semi-lattice structure on $Sub(X)$. We will show that $Sub(X)$ is in fact a Heyting algebra in the next section, after we cover quantification.

The top element is given by a subobject $\mathrm{id} : X \to X$; bottom element is the initial morphism $\perp : 0 \to X$. Meets are given by pullbacks in the following way. Let $m : M \to X, m' : M' \to X$. Then their meet is given by a morphism $m \wedge m' : m^*(M') \to X$, obtained as the composition of $m^*(m')$ with m as shown on the diagram below

$$\begin{tikzcd}
m^*(M') \arrow[r] \arrow[d, hook, "m^*(m')"'] \arrow[dr, dotted, "m\wedge m'"] & M' \arrow[d, hook, "m'"] \\
M \arrow[r, hook, "m"'] & X
\end{tikzcd}$$

Because a pullback of a monomorphism is a monomorphism, the map $m \circ m^*(m')$ is a monomorphism, hence, determines a subobject of X. Clearly, $m \wedge m'$ factors through both m and m'. Thus, $m \wedge m' \leq m, m'$; to show that $m \wedge m'$ is the greatest lower bound it suffices to consider another subobject $n : N \to X$ that factors through m and m', e.g. $n = m' \circ k = m \circ h$.

$$\begin{tikzcd}
N \arrow[drr, bend left, "k"] \arrow[ddr, bend right, "h"'] & & \\
& m^*(M') \arrow[r] \arrow[d, hook, "m^*(m')"'] \arrow[dr, dotted, "m\wedge m'"] & M' \arrow[d, hook, "m'"] \\
& M \arrow[r, hook, "m"'] & X
\end{tikzcd}$$

By the universal property of pullbacks, there is a morphism $N \to m^*(M')$ making the diagram commute; hence, n factors through $m \wedge m'$.

In the rest of this section we continue with the description of subobject in Asm, before moving to arithmetic and completeness theorem. First of all, we give explicit definitions for quantifiers in Asm, and verify the adjunction between existential quantification and substitution, and between substitution and universal quantification. After that we define Heyting implication in every $Sub(X)$ through universal quantification, and verify that it is right adjoint to conjunction. In sections 2.3.4 and 2.3.5 we give concrete specifications for disjunction and equality in Asm; we also show that assemblies are $\neg\neg$-separated. Finally, in section 2.3.6 we prove the completeness result, which says that a first-order formula of natural numbers is valid in Asm iff it is Kleene realizable.

In the rest of this section, each concretely-presented definition of a subobject should be taken as a definition of an assembly in the sense of definition 4. This convention is adopted for the reasons of simplicity of the presentation. To obtain a regular assembly from an assembly*, remove all the elements from the carrier set that are not realizable.

2.3.2 Quantification

We define quantification as an adjoint to substitution

$$\exists_f \dashv f^* \dashv \forall_f$$

for a morphism $f : Y \to X$. Below we give an explicit definition of $\forall_f$.

Suppose that N is a subobject of Y in a "canonical" form, i.e. $N \subseteq Y$ and

$$\langle a, b\rangle \Vdash_N y \iff a \in N_y, b \Vdash_Y y$$

Then $\forall_f(N)$ is defined as follows.

Definition 7. *The carrier set of* $\forall_f(N)$ *is*

$$\{x \in X \mid \forall y \in Y (f(y) = x \to y \in N)\}$$

and the realizability relation is given by

$$\langle e, d\rangle \Vdash_{\forall_f(N)} x \iff \begin{cases} d \Vdash_X x \\ \forall y[f(y) = x \to (\forall a.(a \Vdash_Y y \to ea \Vdash_N y))] \end{cases}$$

i.e. $e \in \bigcap_{f(y)=x}(\|y\|_Y \implies \|y\|_N)$. *Then the inclusion* $\forall_f(N) \to X$ *is tracked by* p_2.

Proposition 8. *Given* $f : Y \to X$, $\forall_f$ *is a right adjoint of* f^*.

Proof. Suppose we are given subobjects $M \in Sub(X)$ and $N \in Sub(Y)$, both in canonical form.

Given $g : f^*(M) \to N$ (in $Sub(Y)$), construct $h : M \to \forall_f(N)$. As a reminder, we reproduce the definition of f^*:

$$f^*(M) = \{(y, x) \mid x \in M, f(y) = x\}$$
$$\langle a, b\rangle \Vdash_{f^*(M)} (y, x) \iff a \Vdash_Y y \text{ and } b \Vdash_M x$$

And the arrow $f^*(M) \to Y$ is given by the first projection. Since π_1 factors through $N \to Y$ via g, we obtain an equation fully characterizing g from the canonical form of N

$$n(g(z)) = g(z) = \pi_1(z)$$

i.e. $g(y, x) = y$.

We define $h : M \to \forall_f(N)$ by $h(x) = x$. Assume that $x \in M$, then for all $y \in f^{-1}(x)$, $(y, x) \in f^*(M)$. Hence, for all $y \in f^{-1}(x)$, $g(y, x) = y \in N$. But that means exactly that $x \in \forall_f(N)$. Thus h is well-defined on the level of sets.

What could be a possible term tracking h? Suppose $b \Vdash_M x$, and $a \Vdash_Y y$ for some $y \in f^{-1}(x)$. Then $\langle a, b\rangle \Vdash_{f^*(M)} (y, x)$. Hence, $\underline{g}\langle a, b\rangle \Vdash_N g(y, x) = y$. Thus

$$\lambda a.\underline{g}\langle a, b'\rangle \in \bigcap_{y \in f^{-1}(x)} (\|y\|_Y \implies \|y\|_N)$$

It follow that $\lambda x.\langle \lambda a.\underline{g}\langle a, x\rangle, \mathsf{p}_2 x\rangle$ tracks h.

In the other direction, suppose we are given $h : M \to \forall_f(N)$, and we want to construct $g : f^*(M) \to N$. Once again, because M is in the canonical form and the inclusion of M into X factors through $\forall_f(N)$, the function h is fully characterized by $h(x) = x$.

We define $g(y, x) = y$. If $(y, x) \in f^*(M)$, then $f(y) = x$ and $x \in M$. Hence $h(x) = x \in \forall_f(N)$, which means that $\forall y' \in f^{-1}(x).y' \in N$. In particular, $y \in N$.

What could be the realizer for g? Suppose $a \Vdash_{f^*(M)} (y, x)$, then $\mathsf{p}_1 a \Vdash_Y y$ and $\mathsf{p}_2 a \Vdash_M x$.

Hence $\underline{h}(\mathsf{p}_2 a) \Vdash_{\forall_f(N)} h(x) = x$, i.e.

$$\begin{cases} \mathsf{p}_2(\underline{h}(\mathsf{p}_2 a)) \Vdash_X x \\ \mathsf{p}_1(\underline{h}(\mathsf{p}_2 a)) \text{ is a term } e \text{ s.t. } \forall y' \in f^{-1}(x), b \Vdash_Y y' \to eb \Vdash_N y' \end{cases}$$

Specifically, $f(y) = x$ and $\mathsf{p}_1 a \Vdash_Y y$, thus $e(\mathsf{p}_1 a) \Vdash_N y$. It follows that g is tracked by

$$\lambda a.\mathsf{p}_1(\underline{h}(\mathsf{p}_2 a))(\mathsf{p}_1 a)$$

□

A similar construction yields the definition of a left adjoint to substitution. Suppose, once again, that $f : Y \to X$ is a morphism of assemblies and N is a subobject of Y.

Definition 9. *The object $\exists_f(N)$ is defined to be a carrier set*

$$\{x \in X \mid \exists y \in Y(f(y) = x \wedge y \in N)\}$$

with the realizability relation

$$\langle e, d\rangle \Vdash_{\exists_f(N)} x \iff \begin{cases} d \Vdash_X x \\ e \Vdash_N y \textit{ for some } y \in N, \textit{ s.t. } f(y) = x \end{cases}$$

The inclusion $\exists_f(N) \hookrightarrow X$ is tracked by p_2.

Proposition 10. *$\exists_f$ is a left adjoint to f^**

Proof. Given $g : M \to f^*(N)$ (which is tracked by a p.r. $\underline{g}$), we construct $h : \exists_f(M) \to N$.

$$
\begin{tikzcd}
M \arrow[r, "g"] \arrow[dr, "m"'] & f^*(N) \arrow[r] \arrow[d, "\pi_1"] & N \arrow[d, "n"] & \exists_f(M) \arrow[l, dotted, "h"'] \arrow[dl] \\
& Y \arrow[r, "f"'] & X &
\end{tikzcd}
$$

We define $h(x) = x$. If $x \in \exists_f(M)$, then there is $y \in M, f(y) = x$. Then $g(y) \in f^*(N)$ is a pair (y', x') such that $f(y') = x'$ and $x' \in N$. But by the commutativity of the diagram above,

$$y = \pi_1(g(y))$$

Hence $y' = y$ and $x' = x$. This shows that h is well-defined.

If $\langle e, d\rangle \Vdash_{\exists_f(M)} x$, then $e \Vdash_M y$ for some $y \in M$, s.t. $f(y) = x$. Then $\underline{g} \cdot e \Vdash_{f^*(N)} g(y)$. But then $\pi_2(g(y)) = x$ (by the same argument as above), and, hence $\mathsf{p}_2(\underline{g} \cdot e) \Vdash_N x$.

For the other direction suppose we are given $h : \exists_f(M) \to N$. If we assume that all subobjects are given canonically, then h is just an identity function, tracked by some p.r. $\underline{h}$.

We then define

$$g(y) = (y, f(y))$$

By definition, $f(y) \in \exists_f(M)$, so $h(f(y)) = f(y) \in N$ and therefore g is well-defined.

If $\langle a, b\rangle \Vdash_M y$, then $b \Vdash_Y y$ and we need to find a realizer for $f(y)$ in N. Since f is tracked by some $\underline{f}$, we get $\underline{f} \cdot b \Vdash_X f(y)$. Then $\langle\langle a, b\rangle, \underline{f} \cdot b\rangle \Vdash_{\exists_f(M)} f(y)$, and hence $\underline{h} \cdot \langle\langle a, b\rangle, \underline{f} \cdot b\rangle \Vdash_N f(y)$. In summary, g is tracked by

$$\lambda x.\langle \mathsf{p}_2 x, \underline{h} \cdot \langle x, \underline{f}(\mathsf{p}_2 x)\rangle\rangle$$

$\square$

2.3.3 Implication and negation

The Heyting implication $m_o \Rightarrow m_1$ can be derived from quantification and substitution. Specifically, we put

$$m_0 \Rightarrow m_1 = \forall_{m_0} m_0^*(m_1)$$

Below we will prove that this definition satisfies the adjunction property.

Lemma 11. *Let c, m_0, m_1 be subobjects of X. Then*

$$\frac{c \leq m_0 \Rightarrow m_1}{\overline{c \wedge m_0 \leq m_1}}$$

Proof. ($\Rightarrow$). Suppose $c \leq m_0 \Rightarrow m_1$. Then,

$$\frac{c \leq m_0 \Rightarrow m_1 = \forall_{m_0} m_0^*(m_1)}{\overline{m_0^*(c) \leq m_0^*(m_1)}} \text{ Adjunction } (m_0)^* \vdash \forall_{m_0}$$

i.e. $m_0^*(c)$ factors through $m_0^*(m_1)$:

$$\begin{tikzcd}
m_0^*(C) \arrow[r, "l"] \arrow[dr, "m_0^*(c)"'] & m_0^*(M_1) \arrow[d, "m_0^*(m_1)"] \\
& M_0
\end{tikzcd}$$

But then $m_0 \wedge c = m_0 \circ m_0^*(c) = m_0 \circ m_0^*(m_1) \circ l = m_0 \wedge m_1 \circ l$, hence $m_0 \wedge c \leq m_0 \wedge m_1 \leq m_1$.

($\Leftarrow$). Suppose $c \wedge m_0 \leq m_1$, i.e. $m_0 \circ m_0^*(c) \leq m_1$. Then

$$\frac{m_0 \circ m_0^*(c) \leq m_1 \qquad m_0 \circ m_0^*(c) \leq m_0}{\overline{m_0 \circ m_0^*(c) \leq m_0 \wedge m_1 = m_0 \circ m_0^*(m_1)}}$$

i.e. $m_0 \circ m_0^*(c) = m_0 \circ m_0^*(m_1) \circ l$ for some l; since m_0 is a mono, we get

$$m_0^*(c) = m_0^*(m_1) \circ l \implies m_0^*(c) \leq m_0^*(m_1) \implies c \leq m_0 \Rightarrow m_1$$

$\square$

Since we've established the explicit interpretation of quantifiers, we can write out the definitions of implication and negation explicitly. If $m : M \to X$ and $n : N \to X$ are subobjects of X (in a canonical form), then

$$\begin{aligned}
M \Rightarrow N &:= \forall_m(m^*(N)) \\
&= \{x \in X \mid \forall x' \in M(m(x') = x \implies x' \in \|m^*(N)\|)\} \\
&= \{x \in X \mid x \in M \implies x \in \|m^*(N)\|\}
\end{aligned}$$

where $\|m^*(N)\|$ is the canonical form of the subobject $m^*(N)$. To remind ourselves,

$$m^*(N) = \{(x, y) \in M \times N \mid m(x) = n(y)\} = \{(x, x) \mid x \in M \cap N\}$$

with the realizability relation

$$b \Vdash_{m^*(N)} (x, y) \iff \mathsf{p}_1 b \Vdash_M x \text{ and } \mathsf{p}_2 b \Vdash_N x$$

This subobject is isomorphic to a subobject in a canonical form $(M \cap N, \Vdash_{M\cap N})$, where $\langle a, b\rangle \Vdash_{M\cap N} x$ iff $a \Vdash_M x$, $b \Vdash_N x$.

Thus

$$M \Rightarrow N = \{x \in X \mid x \in M \implies x \in N\}$$

The realizability relation on that subobject is defined as

$$\langle e, d\rangle \Vdash_{M\Rightarrow N} x \iff \begin{cases} d \Vdash_X x \\ \forall x' \in M(m(x') = x \to \forall a(a \Vdash_M x' \implies ea \Vdash_{m^*(N)} x')) \end{cases}$$

i.e.

$$\langle e, d\rangle \Vdash_{M\Rightarrow N} x \iff \begin{cases} d \Vdash_X x \\ (x \in M \;\&\; a \Vdash_M x) \implies ea \Vdash_{m^*(N)} x \end{cases}$$

Given the realizability relation on the canonical form of $m^*(N)$ it suffices that the first element of the realizer, e in the definition above, satisfy the following property:

$$(x \in M \;\&\; a \Vdash_M x) \implies ea \Vdash_N x$$

Thus linking the classical notion of a realizer for an implication.

Negation, of course, is a specific form of implication, $\neg M := M \Rightarrow \bot$. Concretely, $\neg M = \{x \in X \mid x \in M \implies x \in \emptyset\} = M^c$.

$$\langle e, d\rangle \Vdash_{\neg M} x \iff \begin{cases} d \Vdash_X x \\ (x \in M \wedge a \Vdash_M x) \implies ea \Vdash_\emptyset x \end{cases}$$

As one can note, the second clause in the equation is true trivially – the antecedent requires for x to be M, but realizability relation is defined only for M^c. Thus the negation eliminates almost all the realizability information.

With this insight, let us look at the doubly negated subobjects. The carrier set $\neg\neg M$ is M; however the realizability relation is defined as

$$\langle e, d\rangle \Vdash_{\neg\neg M} x \iff \begin{cases} d \Vdash_X x \\ (x \in \neg M \wedge a \Vdash_{\neg M} x) \implies ea \Vdash_{\emptyset} x \end{cases} \iff d \Vdash_X x$$

Thus doubly-negating a subobject effectively removes the realizability information.

2.3.4 Disjunction

The disjunction/join of two subobjects $M, N \hookrightarrow X$ is given by

$$M \vee N = M \cup N$$

$$\langle a, b\rangle \Vdash_{M\vee N} x \iff \begin{cases} b \Vdash_X x \\ x \in M \wedge \mathsf{p}_1 a = 0 \wedge \mathsf{p}_2 a \Vdash_M x \\ x \in N \wedge \mathsf{p}_1 a = 1 \wedge \mathsf{p}_2 a \Vdash_N x \end{cases}$$

As usual, the inclusion $M \vee N \hookrightarrow X$ is tracked by p_2.

First of all, note that $M, N \leq M \vee N$ are inclusions that are tracked by $\lambda\langle c, d\rangle.\langle\langle i, c\rangle, d\rangle$, with $i = 0, 1$. Secondly, suppose that $M, N \leq P$ are given by morphisms $f : M \to P$ and $g : N \to P$. Then we can define a morphism $h : M \vee N \to P$ as

$$h(x) = \begin{cases} f(x) & \text{if } x \in M \\ g(x) & \text{if } x \in N \end{cases}$$

Note that f and g must agree on $M \cap N$, so h is well defined. Suppose that f and g are tracked by $\underline{f}$ and $\underline{g}$ respectively; then h is tracked by

$$\lambda\langle a, b\rangle.\text{if } \mathsf{p}_1 a = 0 \text{ then } \underline{f}(\mathsf{p}_2 a); \text{ else } \underline{g}(\mathsf{p}_2 a)$$

We leave it to the reader to check that this function is primitive-recursive. Therefore, $M \vee N$ is indeed the join of M and N.

2.3.5 Equality and $\neg\neg$-separation

Recall from section 2.1.1, that the equality relation for a sort A is defined as $\mathrm{Eq}_A := \exists_{\delta_A}(\top)$. Concretely,

$$\mathrm{Eq}_A(x,y) \iff \exists x'.\delta_A(x') = (x',x') = (x,y) \iff x = y$$

and

$$a \Vdash_{\mathrm{Eq}_A} (x,x) \iff \mathsf{p}_2 a \Vdash_A x \wedge \mathsf{p}_1 a \Vdash_{A\times A} (x,x)$$

Clearly, it is isomorphic to an object with the same carrier set, for which the realizability relation is $a \Vdash_{\mathrm{Eq}_A} (x,x) \iff a \Vdash_A x$.

In fact, it is also the case that Eq_A is isomorphic to $A \hookrightarrow A \times A$. Isomorphism is witnessed my maps $x \mapsto (x,x)$ and $(x,x) \mapsto x$, both tracked by $\lambda a.a$.

Now that we know how to interpret first-order logic in assemblies, we can prove that the following statement holds in any assembly X:

$$\forall x : X, y : X(\neg\neg(x = y) \Rightarrow x = y)$$

This amounts to proving $x : X, y : Y \vdash \neg\neg(x = y) \Rightarrow x = y$. Note that

$$\begin{aligned}
[\![\neg\neg(x = y) \Rightarrow (x = y)]\!] =& \{(x,y) \in X \times X \mid (x,y) \in [\![]\!] \Rightarrow (x,y) \in [\![x = y]\!]\}\\
=& \{(x,y) \in X \times X \mid x = y \Rightarrow x = y\}\\
=& X \times X
\end{aligned}$$

The morphism $X \times X \to [\![\neg\neg(x = y) \Rightarrow (x = y)]\!]$ is thus given by $(x,y) \mapsto (x,y)$ and is tracked by $\lambda a.\langle \lambda c.\mathsf{p}_1 a, a\rangle$. Therefore we have

$$x : X, y : X \vdash \top \leq \neg\neg(x = y) \Rightarrow (x = y)$$

This property is called $\neg\neg$-separation, and we will return to it in section 2.4.

2.3.6 Arithmetic and realizability in Asm

One important assembly that we consider is the object $N = \mathbf{N}$ with the realizability relation $n \Vdash_N n$.

Proposition 12. *Together with the maps* $0 : \top \to N$ *(*$0(*) = 0$*) and* $S : N \to N$ *(*$S(n) = n + 1$*),* N *is the natural numbers object in* Asm*.*

Proof. Suppose we have a diagram of the form

$$\begin{tikzcd} 1 \arrow[r, "0"] \arrow[dr, "q"'] & N \arrow[r, "s"] & N \\ & A \arrow[r, "a"] & A \end{tikzcd}$$

where q is tracked by $\underline{q}$, and a is tracked by $\underline{a}$. We have to find an arrow $u : N \to A$ making the diagram above commute. Define

$$u(0) = q(*)$$
$$u(x + 1) = a(u(x))$$

Given a number $n \in \mathbf{N}$ (which realizes $n \in N$), we have $\mathrm{rec}(n; \underline{q} \cdot n; \lambda x, y.\underline{a} \cdot y) \Vdash_A u(n)$.

□

We can "restore" the realizability relation, by looking at the subobjects of N^k in Asm, where N is the natural number object given by the underlying set $\mathbf{N}$ and the realizability predicate $E_N(x) = \{x\}$.

If $\varphi(x_1, \ldots, x_k)$ is a formula of arithmetic, then it corresponds to a subobject $[\![\varphi]\!] \hookrightarrow N^k$. By 6, $[\![\varphi]\!]$ has the canonical form (A, E_A) with $A \subseteq N^k$.

The next theorem establishes a link between Kleene's realizability interpretation and the notion of realizability in Asm.

Theorem 13. *For any formula* $\varphi(x_1, \ldots, x_k)$ *of arithmetic, there are primitive recursive functions* α *and* β *such that for all numbers* $m_1, \ldots, m_k$ *(abbreviated* $\vec{m}$*):*

1. *If* $n \Vdash_{[\![\varphi]\!]} \langle m_1, \ldots, m_k \rangle$*, then* $\alpha(n)\langle \vec{m} \rangle \ \underline{\mathsf{nr}}\ \varphi(\vec{m})$*;*

2. *If* $n \ \underline{\mathsf{nr}}\ \varphi(\vec{m})$*, then* $\beta(n)\langle \vec{m} \rangle \Vdash_{[\![\varphi]\!]} \vec{m}$*.*

Proof. We prove the statement by simultaneous induction on the structure of φ.

Case $\varphi = \bot$. Trivial, as nothing realizes bottom, neither in the number realizability sense, nor in the categorical sense.

Case $\varphi = (s = t)$. Note that $a \Vdash_\varphi \vec{m}$ iff $[\![s(\vec{m})]\!] = [\![t(\vec{m})]\!]$ and $a = \langle \vec{m} \rangle$. Thus put $\alpha(a)\langle \vec{m} \rangle = a$ and $\beta(a)\langle \vec{m} \rangle = \langle \vec{m} \rangle$.

Case $\varphi = A \wedge B$. Then $\langle a, b \rangle \Vdash_{A \wedge B} \langle \vec{m} \rangle$ implies

$$\begin{cases} b \Vdash_N \langle \vec{m} \rangle \text{ i.e., } b = \langle \vec{m} \rangle \\ \mathsf{p}_1 a \Vdash_A \langle m_1, \ldots, m_k \rangle \overset{\text{IH}}{\Longrightarrow} \alpha_1(\mathsf{p}_1 a)\langle \vec{m} \rangle \ \underline{\mathsf{nr}}\ A(\vec{m}) \\ \mathsf{p}_2 a \Vdash_B \langle m_1, \ldots, m_k \rangle \overset{\text{IH}}{\Longrightarrow} \alpha_2(\mathsf{p}_1 a)\langle \vec{m} \rangle \ \underline{\mathsf{nr}}\ B(\vec{m}) \end{cases}$$

Then $\langle \alpha_1(\mathsf{p}_1 a), \alpha_2(\mathsf{p}_2 a) \rangle \ \underline{\mathsf{nr}}\ (A \wedge B)(\vec{m})$. Conversely, suppose

$$\langle c, d \rangle \ \underline{\mathsf{nr}}\ (A \wedge B)(\vec{m}) = A(\vec{m}) \wedge B(\vec{m})$$

Then, by inductive hypothesis, there are p.r. β_1, β_2 such that $\beta_1(c)\langle \vec{m} \rangle \Vdash_A \langle \vec{m} \rangle$, $\beta_2(d)\langle \vec{m} \rangle \Vdash_B \langle \vec{m} \rangle$. Thus, $\langle \langle \beta_1(c)\langle \vec{m} \rangle, \beta_2(d)\langle \vec{m} \rangle \rangle, \langle \vec{m} \rangle \rangle \Vdash_{A \wedge B} \langle \vec{m} \rangle$.

It is easy to obtain the required p.r. functions α and β from the constructions above.

Case $\varphi = A \vee B$. Similarly.

Case $\varphi = A \Rightarrow B$. Then suppose

$$\langle e, d \rangle \Vdash_{A \Rightarrow B} \langle m_1, \ldots, m_k \rangle \iff \begin{cases} d \Vdash_{N^k} \langle m_1, \ldots, m_k \rangle \text{ i.e. } d = \langle m_1, \ldots, m_k \rangle \\ \langle \vec{m} \rangle \in A \wedge a \Vdash_A \langle \vec{m} \rangle \implies ea \Vdash_B \langle \vec{m} \rangle \end{cases}$$

Suppose that $c \ \underline{\mathsf{nr}}\ A(\vec{m})$, then, by inductive hypothesis, $\beta_1(c) \Vdash_A \langle \vec{m} \rangle$. Then $e(\beta_1(c)) \Vdash_B \langle \vec{m} \rangle$, and once again, by inductive hypothesis, $\alpha_2(e(\beta_1(c))) \ \underline{\mathsf{nr}}\ B(\vec{m})$.

On the other hand, suppose $b \ \underline{\mathsf{nr}}\ (A \Rightarrow B)(\vec{m})$, and $\langle \vec{m} \rangle \in A \ \&\ a \Vdash_A \langle \vec{m} \rangle$. Then $b(\alpha_1(a)) \ \underline{\mathsf{nr}}\ B(\vec{m})$ and $\beta_2(b(\alpha_1(a))) \Vdash_B \langle \vec{m} \rangle$.

Case $\varphi = \forall y : \mathbf{N}.A(x, x_1, \ldots, x_k)$. Suppose that

$$\langle e, d \rangle \Vdash_{\forall x.A} \langle m_1, \ldots, m_k \rangle \iff \begin{cases} d = \langle m_1, \ldots, m_k \rangle \\ \forall n, \forall a.a \Vdash_{N^{k+1}} \langle n, m_1, \ldots, m_k \rangle \Rightarrow ea \Vdash_A \langle n, m_1, \ldots, m_k \rangle \end{cases}$$

Note that $a \Vdash_{N^{k+1}} \langle n, m_1, \ldots, m_k \rangle \iff a = \langle n, m_1 \ldots, m_k \rangle$. By inductive hypothesis, $\alpha(e\langle n, \vec{m} \rangle) \ \underline{\mathsf{nr}}\ A(n, \vec{m})$. Hence $\lambda x.\alpha(e\langle x, \vec{m} \rangle) \ \underline{\mathsf{nr}}\ \forall x : \mathbf{N}.A(x, \vec{m})$.

For the other direction, suppose that $l \ \underline{\mathsf{nr}}\ \forall x : \mathbf{N}.A(x, \vec{m})$. Then for all n, $\beta(l \cdot n) \Vdash_A \langle n, \vec{m} \rangle$. Hence

$$\langle \lambda a.\beta(l \cdot \mathsf{p}_1 a), \langle \vec{m} \rangle \rangle \Vdash_{\forall x.A} \langle \vec{m} \rangle$$

Case $\varphi = \exists y : \mathbf{N}.A(x, x_1, \ldots, x_k)$. Similarly.

$\square$

2.4 Effective topos

As we have seen, we can interpret first-order many-sorted logic in the category of assemblies; however we cannot fully interpret higher-order logic in Asm as it is not a topos. To obtain a topos from Asm we must freely adjoin all quotients, by what is called an *ex/reg completion*. Specifically, the effective topos $\mathcal{E}ff$ is defined as following (see e.g. [8] for a more detailed explanation).

The objects of $\mathcal{E}ff$ are pairs $(X, \sim)$, where $\sim$ is a function $X \times X \to \mathcal{P}(\mathbf{N})$, usually written as $x, y \mapsto [x \sim y]$, such that there are p.r. functions s and tr satisfying

1. If $a \in [x \sim y]$, then $\mathsf{s}(a) \in [y \sim x]$;
2. If $a \in [x \sim y]$ and $b \in [y \sim z]$, then $\mathsf{tr}(a, b) \in [x \sim z]$.

Given two objects $(X, \sim_X)$ and $(Y, \sim_Y)$ of the effective topos, a map between them as a relation $F : X \times Y \to \mathcal{P}(\mathbb{N})$, satisfying the following conditions:

(REL) Given $n \in [x \sim_X x']$, $m \in [y \sim_Y y']$ and $p \in F(x, y)$, one can recursively and uniformly find an element $\phi(n, m, p) \in F(x', y')$

(ST) Given $n \in F(x, y)$ one can recursively find $\psi_1(n) \in [x \sim_X x]$ and $\psi_2(n) \in [y \sim_Y y]$

(SV) Given $n \in F(x, y)$ and $m \in F(x, y')$, one can recursively find $\chi(n, m) \in [y \sim_Y y']$

(TOT) Given $n \in [x \sim_X x]$, one can recursively find $\rho(n) \in \bigcup_{y \in Y} F(x, y)$

A morphism between assemblies is a set-level function with a requirement of existence of a p.r. function that sends realizers to realizers. In the effective topos in addition to the "existence" realizers we have realizers for the non-trivial equalities $[x \sim y]$. Some morphisms between the objects of $\mathcal{E}ff$ are induced by having a set-level function together with p.r. functions tracking both "existence" realizers and equivalence realizers.

Definition 14 (Set-level induced morphism)**.** *Given two objects $(X, \sim)$ and $(Y, \approx)$ of $\mathcal{E}ff$, a set-level function $f : X \to Y$ that satisfies the following requirements:*

1. *There is a p.r. φ, s.t. $a \in E_X(x) \implies \varphi(a) \in E_Y(f(x))$;*
2. *There is a p.r. ψ, s.t. $a \in [x \sim y] \implies \psi(a) \in [f(x) \approx f(y)]$;*

induces a morphism $F : (X, \sim) \to (Y, \approx)$ given by

$$F(x, y) = \bigcup_{x' \in X} \{\langle a, b\rangle \mid a \in [x \sim x'], b \in [f(x') \approx y]\}$$

We leave it to the reader to verify that F is indeed a morphism in $\mathcal{E}ff$.

Lemma 15. *Every object $(X, \sim)$ of $\mathcal{E}ff$ is isomorphic to an object $(X', \sim)$ such that $E_{X'}(x)$ is non-empty for every $x \in X'$.*

Proof. Put $X' = \{x \in X \mid E_X(x) \neq \emptyset\}$. The realizability relation on X' is then a restriction of $\sim$ to X'.

The morphism $F : (X', \sim) \to (X, \sim)$ is just the inclusion. Specifically, the relation F is induced by the inclusion $x \mapsto x$:

$$F(x, y) = \bigcup_{x' \in X'} \{\langle a, b\rangle \mid a \in [x \sim x'], b \in [x' \sim y]\}$$

The inverse of F is a morphism morphism $G : (X, \sim) \to (X', \sim)$ defined as

$$G(x, y) = [x \sim y]$$

We will show that $F \circ G \simeq \mathrm{id}_X$ and $G \circ F \simeq \mathrm{id}_{X'}$.

By definition,

$$(F \circ G)(x, z) = \{\langle a, b, c\rangle \mid \exists y \in X', a \in E_{X'}(y), b \in F(x, y), c \in G(y, z)\}$$

which is equivalent to

$$\{\langle a, \langle b_1, b_2\rangle, c\rangle \mid \exists y \in X', a \in E_{X'}(y), \langle b_1, b_2\rangle \in F(x, y), c \in [y \sim z]\}$$

In addition,

$$\langle b_1, b_2\rangle \in F(x,y) \iff \begin{cases} b_1 \in [x \sim x'] & \text{for some } x' \in X' \\ b_2 \in [x' \sim y] \end{cases}$$

Thus, $\langle a, \langle b_1, b_2\rangle, c\rangle \in (F \circ G)(x,z) \implies \mathsf{tr}(\mathsf{tr}(b_1, b_2), c) \in [x \sim z] = \mathrm{id}_X(x,z)$. On the other hand, if $a \in [x \sim z]$, then $\mathsf{tr}(a, \mathsf{s}(a)) \in [x \sim x]$, $\mathsf{tr}(\mathsf{s}(a), a) \in [z \sim z]$. Hence,

$$b = \langle \mathsf{tr}(a, \mathsf{s}(a)), a\rangle \in F(x,z)$$

and

$$\langle \mathsf{tr}(\mathsf{s}(a), a), b, \mathsf{tr}(\mathsf{s}(a), a)\rangle \in (F \circ G)(x,z)$$

Therefore, $F \circ G \simeq \mathrm{id}_X$. Similarly, we can show that $G \circ F \simeq \mathrm{id}_{X'}$.

□

Definition 16. *An object* $(X, \sim)$ *of the effective topos is* $\neg\neg$-separated *if there is a p.r. function* ϕ*, such that if* $x, y \in X$*,* $a \in E_X(x)$*,* $b \in E_X(y)$ *and* $[x \sim y]$ *is nonempty, then* $\phi ab \in [x \sim y]$.

Clearly, assemblies are $\neg\neg$-separated. In fact, the converse holds as well:

Lemma 17. *If* $(X, \sim)$ *is* $\neg\neg$*-separated, then* $(X, \sim)$ *is isomorphic to the assembly* $(X/R, E)$ *where*

- $R = \{(x,y) \mid [x \sim y] \neq \emptyset\}$
- $E([x]) = \bigcup_{x' \in [x]} [x' \sim x']$

Proof. We may suppose that all elements of X "exist" (see lemma 15). Then we define $f : X \to X/R$ by $f(x) = [x] = \{y \mid [x \sim y] \neq \emptyset\}$. Then if $a \in E_X(x) \implies a \in E_{X/R}([x])$ and $a \in [x \sim x'] \implies t(a, s(a)) \in E_{X/R}([x]) = E_{x/R}([x'])$. This function f induces a morphism $F : (X, \sim) \to (X/R, E)$ in Eff.

The morphism going in the other direction $G : (X/R, E) \to (X, \sim)$ is given by

$$G([x], y) = \begin{cases} \emptyset & \text{if } [y \sim x] = \emptyset \\ [y \sim y] & \text{otherwise} \end{cases}$$

We can see that G is a well-defined morphism:

(ST) If $a \in G([x], y)$, then $[x \sim y]$ is non-empty and thus $a \in E_{X/R}([x])$ and $a \in [y \sim y]$.

(REL) If $a \in G([x], y)$ and $b \in E([x])$ and $c \in [y \sim y']$, then $\mathsf{p}_1\langle b, a\rangle \in G([x], y)$ and $t(s(c), c) \in G([x], y')$.

(TL) If $a \in E([x])$, then $a \in [x' \sim x']$ for some x' s.t. $[x \sim x']$ is non-empty; thus $a \in G([x], x')$.

(SV) Let $a \in G([x], y) = [y \sim y]$ and $b \in G([x], y') = [y' \sim y']$, then, because X is $\neg\neg$-separated, there is a p.r. function ϕ such that $\phi ab \in [y \sim y']$ if $[y \sim y']$ is non-empty. However, because $a \in G([x], y)$ implies that $[x \sim y]$ is non-empty, and $b \in G([x], y')$ implies $[x \sim y']$ is non-empty, we know that $[y \sim y']$ is non-empty.

□

Chapter 3

Homotopy theory in the category of assemblies

The goal of this chapter is to show how it is possible to view assemblies as spaces, and understand certain constructions in topological terms.

First of all, in this chapter we establish the structure of *path object category* [1, 2] on the category of assemblies. The developments in that section follow the description of homotopical notions in the effective topos, which were first discovered and described in [7]. In the next section we describe the notions of a path, a path space, and a notion of path contraction onto the path's endpoint.

In the consecutive section, we describe, purely in terms of path object categories, notions of *homotopy* and *strong deformation retract* (following [1, 2]); we also derive a slightly more "advanced" path contraction, which was previously given in [7] specific to the effective topos.

In section 3.3 we generalize a notion of a Hurewicz fibration as given in [7] to an arbitrary nice path object category, and show, following [1] that there is a functorial factorization system in a path object category, in which every map is factored as a strong deformation retract followed by a Hurewicz fibration.

Finally, in section 3.4 we describe *modest sets* as discrete "spaces" and spell out the fact that discrete objects form a reflexive subcategory in *Asm*. The material is that subsection follows [7, 8].

3.1 Path object categories

Path object categories, originally defined in [1] and further refined in [2], are settings for doing basic homotopy theory and for producing split models of Martin-Löf type theory. In this section we recall the axioms of a path object category and show that the category of assemblies is an instance of a path object category.

Definition 18 ((Nice) path object category [2])**.** *Given a finitely complete category $\mathcal{C}$, a* nice path object category *is a tuple $(\mathcal{C}, P, s, t, c, \tau, *)$ such that*

- $P : \mathcal{C} \to \mathcal{C}$ *is a pullback-preserving functor, called the path object functor*
- $s, t : P \Rightarrow \mathrm{id}$
- $c : \mathrm{id} \Rightarrow P$
- $\tau : P \Rightarrow P$
- $* : PX \times_X PX \to PX$*, natural in* X

where the $PX \times_X PX$ is the pullback

$$\begin{tikzcd} PX \times_X PX \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & PX \arrow[d, "s_X"] \\ PX \arrow[r, "t_X"] & X \end{tikzcd}$$

1. *For all X,* $X \underset{s_X}{\overset{t_X}{\leftleftarrows}} \xrightarrow{c_X} PX \xleftarrow{*} PX \times_X PX$ *is an* internal category*, with τ being identity-on-objects involution of the internal category. Specifically*

$$\begin{array}{ll} \tau_X \circ \tau_X = \mathrm{id}_X & \tau_X \circ c_X = c_X \\ s_X \circ \tau_X = t_X & \tau_X \circ * = * \circ \langle \tau\pi_2, \tau\pi_1 \rangle \\ t_X \circ \tau_X = s_X & \end{array}$$

2. $\mathcal{C}$ *has nice constant paths, that is*

$$P1 \simeq 1$$

3. $\mathcal{C}$ has nice path contraction, that is, there is a natural transformation

$$\eta : PP \Rightarrow P$$

such that

$$\begin{array}{cc} s_{PX} \circ \eta_X = \mathrm{id}_{PX} & P(s_X) \circ \eta_X = \mathrm{id}_{PX} \\ t_{PX} \circ \eta_X = c_X \circ t_X & P(t_X) \circ \eta_X = c_X \circ t_X \\ \eta_X \circ c_X = c_{PX} \circ c_X & \end{array}$$

A more intuitive explanation of the components of this definitino shall follow below, as we discuss specifics of this construction in Asm.

3.1.1 Intervals and path objects

An *interval* of length n is an assembly $I_n = \{0, \ldots, n\}$ with the realizability relation $m \Vdash_{I_n} i$ iff $m = i$ or $m = i + 1$. A *path* of length n (n-path), in an object X, is a morphism $p : I_n \to X$. An exponential X^{I_n} is thus a collection of n-paths in X.

It is tempting to define a path object over X as a sum $\Sigma_{n\geq 0} X^{I_n}$. In this case we would have something like a Moore-path object, where paths have different lengths. To obtain a nicer path object we want to quotient this collection by an equivalence relation.

Definition 19. *A map $\sigma : I_n \to I_m$ is order and endpoint preserving iff it is order preserving and satisfies $\sigma(0) = 0$ and $\sigma(n) = m$.*

Every such map is surjective; for, suppose that $j \in I_m$ is not in the image of σ and σ is tracked by φ. But then $\sigma(i) < j$ and $\sigma(i + 1) > j$ for some i. Since $i + 1$ realizes both i and $i + 1$, it must be the case that $\varphi(i + 1)$ realizes both $\sigma(i)$ and $\sigma(i + 1)$, but they should not a realizer in common. Hence an order and endpoint preserving map $\sigma : I_n \to I_m$ exists only if $n \geq m$.

We can now give a definition of a path object.

Definition 20. *Given an assembly (X, E) we can construct a path object $P(X, E)$ (which we write as $P(X)$ when it is unambiguous) the underlying set of which is a quotient of $\{(n, f) \mid f : I_n \to X\}$ by the relation $\sim$. We say that $(n, f) \sim (m, g)$ if*

1. *$n \geq m$ and there is an order and endpoint preserving map $\sigma : I_n \to I_m$ such that $f = g\sigma$; or*

2. *$m \geq n$ and there is an order and endpoint preserving map $\sigma : I_m \to I_n$ such that $g = f\sigma$.*

The realizability relation on $P(X)$ is given by

$$E_{P(X)}([(n,f)]) = \bigcup_{(m,g)\in[(n,f)]} \{\langle m, b\rangle \mid b \Vdash_{X^{I_m}} g\}$$

Sometimes we will denote the equivalence class $[(n,f)]$ by $[n,f]$. We write $(n,p) \rightsquigarrow_\sigma (m,q)$ when $\sigma : I_n \to I_m$ is an order and endpoint preserving map, and $p = q\sigma$; in particular $(n,p) \rightsquigarrow_\sigma (m,q)$ implies $(n,p) \sim (m,q)$.

The construction in definition 20 extends to a pullback-preserving endofunctor $P : Asm \to Asm$. Indeed, suppose that $f : X \to Y$ is a map of assemblies, tracked by $\underline{f}$. Then the map $P(f) : P(X) \to P(Y)$ is defined by

$$[n,p] \mapsto [n, f \circ p]$$

To see that this is well defined, assume $(n,p) \rightsquigarrow_\sigma (m,q)$. But then $fp = fq\sigma$, hence $P(f)([n,p]) = P(f)([m,q])$. The map $P(f)$ is tracked by $\lambda\langle m,b\rangle.\langle m, \lambda x.\underline{f}(b(x))\rangle$. It is clear that P obeys the functorial laws.

To show that P preserves pullbacks, i.e. that for each pullback square

$$\begin{array}{ccc} Z \times_Y X & \xrightarrow{\pi_2} & X \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & Y \end{array}$$

we have $P(Z \times_Y X) \simeq P(Z) \times_{P(Y)} P(X)$. It suffices to show that for each object A and morphisms $h : A \to PX$ and $k : A \to PZ$, such that $P(g) \circ k = P(f) \circ h$, there is a unique morphism $A \to P(Z \times_Y X)$, making the diagram below commute.

$$\begin{tikzcd}
A \arrow[drr, bend left, "h"] \arrow[dr, dashed, "r"] \arrow[ddr, bend right, "k"'] & & \\
& P(Z \times_Y X) \arrow[r, "P(\pi_2)"] \arrow[d, "P(\pi_1)"] & PX \arrow[d, "P(f)"] \\
& PZ \arrow[r, "P(g)"] & PY
\end{tikzcd}$$

Suppose that $a \in A$ and $h(a) = [n, p] \in PX, k(a) = [m, q] \in PZ$. Furthermore, $P(f)(h(a)) = P(g)(k(a))$, hence, $(n, fp) \sim (m, gq)$. Without loss of generality, suppose $(n, fp) \rightsquigarrow_\sigma (m, gq)$.

The put $r(a) = [(n, \theta)]$, where $\theta(x) = (p(x), q(\sigma(x)))$. Clearly, $\theta(x) \in Z \times_Y X$, as $f(p(x)) = g(q(\sigma(x))$.

For each path $[n, p]$ we define its source and target by natural transformations

$$s_X([n, p]) = p(0)$$
$$t_X([n, p]) = p(n)$$

We can see that s, t are well-defined: suppose $(n, p) \rightsquigarrow_\sigma (m, q)$, then, since σ is endpoint-preserving, $p(0) = q(\sigma(0)) = q(0)$; similarly for t.

In addition, s_X is tracked by $\lambda\langle m, b\rangle.b0$ and t_X is tracked by $\lambda\langle m, b\rangle.bm$.

To show that s and t are natural, consider a diagram

$$\begin{tikzcd}
PX \arrow[r, "P(f)"] \arrow[d, "s_X"'] & PY \arrow[d, "s_Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Given $[n, p] \in PX$, $s_Y(P(f)(n, p)) = f(p(0)) = f(s_Y(n, p))$, hence the diagram above commutes; similarly for t.

For each point x_0 we have a *constant path* $c_X(x_0) = [0, i \mapsto x_0]$. The map c_X is tracked by $\lambda a.\langle 0, \lambda i.a\rangle$. It is a matter of calculation to see that c is natural in X.

Each path $[(n, p)]$ can be reversed, to obtain a path

$$\widetilde{[(n, f)]} = [(n, i \mapsto f(n - i))]$$

We can see that path reversal is well-defined, for if $(n, f) \rightsquigarrow_\sigma (m, g)$, then the map $\sigma'(x) = m - \sigma(n - x)$ is order and endpoint preserving, and $\widetilde{(n, f)} \rightsquigarrow_{\sigma'} \widetilde{(m, g)}$.

The map $\widetilde{-}$ is tracked by $\lambda\langle m, b\rangle.\langle m, \lambda i.b(m - i)\rangle$.

Finally, we are to describe path composition. Given an n-path (n, p) and an m-path (m, q), s.t. $p(n) = q(0)$, we define $(n, p) * (m, q)$ to be $(n+m, p \star q)$, where

$$(p \star q)(i) = \begin{cases} p(i) & \text{if } i \leq n \\ q(i - n) & \text{if } i \geq n \end{cases}$$

To verify that this operation extends to a map $PX \times_X PX \to PX$, we must show that it is well-defined w.r.t. the equivalence classes in PX. So suppose that $(n, p) \sim (n', p')$ and $(m, q) \sim (m', q')$; we are to show that $(n, p) * (m, q) \sim (n', p') * (m', q')$. We shall treat only the most indirect case: $n \leq n'$ and $m' \leq m$, in other words $(n', p') \rightsquigarrow_\sigma (n, p)$ and $(m, q) \rightsquigarrow_\delta (m', p')$. We are going to show that $(n + m, p \star q) \leftsquigarrow (n' + m, p' \star q) \rightsquigarrow (n' + m', p' \star q)$. See fig. 3.1.

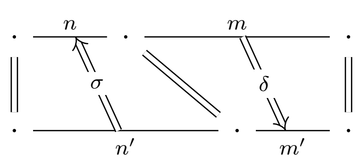

Figure 3.1: $(n', p') \rightsquigarrow_\sigma (n, p)$ and $(m, q) \rightsquigarrow_\delta (m', p')$

We do so in several steps.

Step 1: $(n' + m, p' \star q) \rightsquigarrow_{\sigma_0} (n + m, p \star q)$. We define an endpoint and order-preserving map σ_0

$$\sigma_0(i) = \begin{cases} \sigma(i) & \text{if } i \leq n' \\ n + (i - n') & \text{if } i > n' \end{cases}$$

To see that σ_0 is realizable, suppose we have $x \Vdash_{I_{n'+m}} i$; we can decide whether $x \leq n'$. If it is the case, then return $\underline{\sigma} x$ (where $\underline{\sigma}$ tracks σ), otherwise return $n + (x - n')$. It is straightforward to check that such computable procedure tracks σ_0. It remains to check the following:

$$(p \star q)(\sigma_0(i)) = \begin{cases} (p \star q)(\sigma(i)) & \text{if } i \leq n' \text{ (hence, } \sigma(i) \leq n) \\ (p \star q)(n + (i - n')) & \text{otherwise} \end{cases}$$
$$= \begin{cases} p(\sigma(i)) = p'(i) = (p' \star q)(i) & \text{if } i \leq n' \\ q(i - n') = (p' \star q)(i) & \text{otherwise} \end{cases}$$

Step 2: $(n' + m, p' \star q) \rightsquigarrow_{\sigma_1} (n' + m', p' \star q')$. We define an endpoint and order-preserving map σ_1

$$\sigma_1(i) = \begin{cases} i & \text{if } i \leq n' \\ n' + \delta(i - n') & \text{if } i > n' \end{cases}$$

We can see that

$$(p' \star q')(\sigma_1(i)) = \begin{cases} (p' \star q')(i) & \text{if } i \leq n' \\ (p' \star q')(n' + \delta(i - n') & \text{if } i \geq n' \end{cases}$$
$$= \begin{cases} p'(i) = (p' \star q)(i) & \text{if } i \leq n' \\ q'(\delta(i - n')) = q(i - n') = (p' \star q)(i - n') & \text{if } i \geq n' \end{cases}$$

3.1.2 Internal groupoid

To check that $X \underset{s_X}{\overset{t_X}{\leftleftarrows}} \xrightarrow{c_X} PX \xleftarrow{*} PX \times_X PX$ is an internal categroy, we must confirm several axioms.

First of all, we must verify the identity laws

$$\begin{tikzcd} X \arrow[r, "c_X"] \arrow[dr, equal] & PX \arrow[d, "s_X"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "c_X"] \arrow[dr, equal] & PX \arrow[d, "t_X"] \\ & X \end{tikzcd}$$

Which amounts to checking that $s_X(c_X(x_0)) = x_0 = t_X(c_X(x_0))$; that it holds is clear from the definitions.

Next, we must check that composition respects sources/targets:

$$\begin{tikzcd} PX \times_X PX \arrow[r, "*"] \arrow[d, "\pi_1"] & PX \arrow[d, "s_X"] \\ PX \arrow[r, "s_X"] & X \end{tikzcd} \qquad \begin{tikzcd} PX \times_X PX \arrow[r, "*"] \arrow[d, "\pi_2"] & PX \arrow[d, "t_X"] \\ PX \arrow[r, "t_X"] & X \end{tikzcd}$$

That is, given $[n,p]$ and $[m,q]$, such that $t([n,p]) = s([m,q])$, we have $s([n,p]*[m,q]) = s([n,p])$ and $t([n,p]*[m,q]) = t([m,q])$. It is straightforward to check that $s([n,p]*[m,q]) = (p \star q)(0) = p(0) = s([n,p])$ and $t([n,p]*[m,q]) = (p \star q)(n+m) = q(m)$.

Furthermore, we have to check that composition respects constant paths (from right and from left)

$$\begin{tikzcd} PX \arrow[r, "\langle \mathrm{id}, c_X t_X \rangle"] \arrow[dr, equal] & PX \times_X PX \arrow[d, "*"] & PX \arrow[l, "\langle c_X s_X, \mathrm{id} \rangle"'] \arrow[dl, equal] \\ & PX & \end{tikzcd}$$

Once again, it is a straightforward calculation. For instance, the right identity law

$$(* \circ \langle \mathrm{id}\, c_X t_X \rangle)([n,p]) = [n,p] * c_X(t_X(n,p)) = [n, p \star (i \mapsto p(n))] = [n,p]$$

Finally, we have to check that internal composition is associative; that is, the following diagram commutes

$$\begin{tikzcd} PX \times_X PX \times_X PX \arrow[r, "\langle *, \mathrm{id} \rangle"] \arrow[d, "\langle \mathrm{id}, * \rangle"] & PX \times_X PX \arrow[d, "*"] \\ PX \times_X PX \arrow[r, "*"] & PX \end{tikzcd}$$

Which follows from the fact that the composition of n-paths $\star$ is associative.

To see see that $\tilde{\cdot}$ is internal path reversal, we must verify the following axioms:

1. $\tilde{\cdot} \circ \tilde{\cdot} = \mathrm{id}_{P(X)}$

$$\widetilde{\widetilde{(n,p)}} = (n, \widetilde{i \mapsto p(n-i)}) = (n, i \mapsto p(n-(n-i))) = (n,p)$$

2. $s_X \circ \tilde{\cdot} = t_X$ & $t_X \circ \tilde{\cdot} = s_X$

$$s_X(\widetilde{(n,p)}) = s_X(n, i \mapsto p(n-i)) = p(n) = t_X(n,p)$$

3. $\tilde{\cdot} \circ c_X = c_X$

$$\widetilde{c_X(x_0)} = (0, i \mapsto x_0) = c_X$$

4. $\tilde{\cdot} \circ * = * \circ \langle \tilde{\cdot} \circ \pi_2, \tilde{\cdot} \circ \pi_1 \rangle$

$$\begin{aligned}
(\widetilde{(n,p) * (m,q)})(i) &= (\widetilde{n+m, p \star q}))(i) = (p \star q)(n+m-i) \\
&= \begin{cases} p(n+m-i) = \widetilde{(n,p)}(i-m) & \text{if } i \geq m \\ q(m-i) = \widetilde{(m,q)}(i) & \text{if } i \leq m \end{cases} \\
&= (\widetilde{(m,q)} * \widetilde{(n,p)})(i)
\end{aligned}$$

3.1.3 Nice constant paths and nice path contraction

Constant paths. The path object functor supports *nice constant paths*, that is

$$P1 \simeq 1$$

To see that this holds, it is sufficient to provide a map $1 \to P1$, which is the left inverse to the terminal map $\top : P1 \to 1$. We claim that $c_1 \circ \top = \mathrm{id}_{P1}$, we can think of this as $P1$ being "contractible". Consider an n-path $p : I_n \to 1$. Clearly, p has the form $p(i) = *$. Then, consider an order and endpoint preserving map map $\sigma : I_n \to I_0$ defined by $i \mapsto 0$ and tracked by $\lambda x.0$; it is then the case that $(n,p) \rightsquigarrow_\sigma c_1(*)$.

Path contraction. The goal of the rest of this subsection is to provide a natural transformation $\eta : P \Rightarrow PP$, that contracts a path onto its target; that is, η is subject to the following axioms:

$$\begin{tikzcd}
PX \arrow[r, "\eta_X"] \arrow[d, "\eta_X"'] \arrow[dr, equal] & PPX \arrow[d, "s_{PX}"] \\
PPX \arrow[r, "P(s_X)"'] & PX
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "c_X"] \arrow[d, "c_X"'] & PX \arrow[d, "\eta_X"] \\
PX \arrow[r, "c_{PX}"'] & PPX
\end{tikzcd}$$

$$\begin{tikzcd}
PX \arrow[r, "\eta_X"] \arrow[d, "t_X"'] & PPX \arrow[d, "t_{PX}"] \\
X \arrow[r, "c_X"'] & PX
\end{tikzcd}
\qquad
\begin{tikzcd}
PX \arrow[r, "\eta_X"] \arrow[d, "t_X"'] & PPX \arrow[d, "P(t_X)"] \\
X \arrow[r, "c_X"'] & PX
\end{tikzcd}$$

First of all, we define a map $\alpha_n : X^{(I_n)} \to (X^{(I_n)})^{I_n}$ that satisfies the properties above. Then we show that this map extends to $\eta_X : PX \to PPX$.

To give a map $I_n \to (X^{I_n})$ is to give a sequence $(p_0, \ldots, p_n)$, such that each p_i, p_{i+1} has a realizer in common in X^{I_n}. We put

$$p_i(j) = \begin{cases} p(i) & \text{if } j \leq i \\ p(j) & \text{if } j \geq i \end{cases}$$

Then, in particular, $p_0 = p$ and p_n is the constant path over $p(n)$. For each i, we can effectively calculate the realizers for p_i. Suppose that b is a realizer for p, then we define a function $i \mapsto \underline{p_i}$ by induction, with the base case $\underline{p_0} = b$.

$$\underline{p_{i+1}} = \lambda x. \begin{cases} \text{if } x \leq i+1 \text{ then} & \underline{p_i}(i+1) \\ \text{if } x > i+1 \text{ then} & \underline{p_i}(x) \end{cases}$$

We denote the effective function assigning $\underline{p_i}$ to i as $\phi(b, i)$.

Proposition 21. *If $\underline{p_i}$ tracks p_i, then $\underline{p_{i+1}}$ tracks p_{i+1}.*

Proof. We distinguish cases

Case $j \leq i$. Then $p_{i+1}(j) = p(i+1) = p_{i+1}(i+1) = p_i(i+1)$.

- $\underline{p_{i+1}}j = \underline{p_i}(i+1) \Vdash_X p_i(i+1) = p_{i+1}(i+1)$
- $\underline{p_{i+1}}(j+1) = \underline{p_i}(i+1) \Vdash_X p_i(i+1) = p_{i+1}(i+1)$

Case $j > i$. Then $p_{i+1}(j) = p(j) = p_i(j)$.

- $\underline{p_{i+1}}j = \underline{p_i}(j) \Vdash_X p_i(j) = p_{i+1}(j)$
- $\underline{p_{i+1}}(j+1) = \underline{p_i}(j+1) \Vdash_X p_i(j) = p_{i+1}(j)$

□

Furthermore, $\underline{p_{i+1}}$ is a realizer both for p_i and p_{i+1}

Proposition 22. *$\underline{p_{i+1}}$ tracks p_i.*

Proof. Again, we distinguish two cases

Case $j \leq i$. Then $p_i(j) = p(i) = p_i(i)$.

- $\underline{p_{i+1}}j = \underline{p_i}(i+1) \Vdash_X p_i(i)$
- $\underline{p_{i+1}}(j+1) = \underline{p_i}(i+1) \Vdash_X p_i(i)$

Case $j > i$. Then

- $\underline{p_{i+1}}j = \underline{p_i}(j) \Vdash_X p_i(j)$
- $\underline{p_{i+1}}(j+1) = \underline{p_i}(j+1) \Vdash_X p_i(j)$

□

Hence, the assignment $p \mapsto (i \mapsto p_i)$ defines a morphism $\alpha_n : X^{I_n} \to (X^{I_n})^{I_n}$, which is tracked by $\lambda b.\lambda i.\phi(b,i)$.

It remains to show that α_n extends to a well-defined function $PX \to PPX$, that is, if $(n,p) \rightsquigarrow_\sigma (m,q)$, then $(n, i \mapsto [n,p_i]) \sim (m, i \mapsto [m, q_i])$. Note that any endpoint order preserving map $\sigma : I_n \to I_m$ can be decomposed into "degeneracy" maps $s_k : I_{e+1} \to I_e$ that "hits" k two times[1]:

$$s_k[0, \dots, e+1] = [0, \dots, k, k, \dots, e]$$

If given $(n+1, p) \rightsquigarrow_{s_k} (n,q)$ we manage to show that $(n+1, i \mapsto [n+1, p_i]) \sim (n, i \mapsto [n, q_i])$, then we can do this for an arbitrary σ, by pasting together results for each s_k.

So, suppose $p = qs_k$. We will show the following

Proposition 23.

$$\alpha_{n+1}(p) = \langle (qs_k)_0, \dots, (qs_k)_{n+1} \rangle = \langle q_0 s_k, \dots, q_k s_k, q_k s_k, \dots, q_n s_k \rangle$$

(where $\langle q_1, \dots, q_n \rangle = \alpha_n(q)$)

Proof. Case 1: $i \le k \implies (qs_k)_i = q_i s_k$.

$$\begin{aligned}
(qs_k)_i(j) &= \begin{cases} q(s_k(i)) & \text{if } j \le i \le k \\ q(s_k(j)) & \text{if } j > i, i \le k \end{cases} \\
&= \begin{cases} q(i) = q_i(j) = q_i(s_k(j)) & \text{if } j \le i \le k \\ q(s_k(j)) = q(j) = q_i(j) = q_i(s_k(j)) & \text{if } i < j \le k \\ q(s_k(j)) = q(j-1) = q_i(j-1) = q_i(s_k(j)) & \text{if } i \le k < j \end{cases} \\
&= q_i(s_k(j))
\end{aligned}$$

[1] Formally, such map can be denoted as s_k^e, but here we follow a convention of leaving e implicit

Case 2: $i > k \implies (s_k q)_i = s_k q_{i-1}$.

$$
\begin{aligned}
(qs_k)_i(j) &= \begin{cases} q(s_k(i)) & \text{if } j \le i, i > k \\ q(s_k(j)) & \text{if } j \ge i > k \end{cases} \\
&= \begin{cases} q(i-1) = q_{i-1}(j) = q_{i-1}(s_k(j)) & \text{if } j \le k < i \\ q(i-1) = q_{i-1}(j-1) = q_{i-1}(s_k(j)) & \text{if } k < j \le i \\ q(j-1) = q_{i-1}(j-1) = q_{i-1}(s_k(j)) & \text{if } j \ge i > k \end{cases} \\
&= q_{i-1}(s_k(j))
\end{aligned}
$$

□

From proposition 23 it follows that

$$(i \mapsto [n+1, p_i]) = (i \mapsto [n+1, (qs_k)_i]) = (i \mapsto [n, q_i]) \circ s_k$$

hence, the extension of α to PX is well-defined and we obtain a morphism $\eta_X : PX \to PPX$.

It is then straightforward to verify that path contraction η satisfies the required laws.

3.2 Homotopies and "extended" path contractions

Homotopies and deformation retracts. Given maps $f, g : X \to Y$, a homotopy θ between f and g (written $\theta : f \to g$) is a map $X \to PY$, such that $s_Y \circ \theta = f$ and $t_Y \circ \theta = g$.

$$
\begin{tikzcd}
 & Y \\
X \arrow[ur, "f"] \arrow[r, "\theta"] \arrow[dr, "g"'] & PY \arrow[u, "s_Y"'] \arrow[d, "t_Y"] \\
 & Y
\end{tikzcd}
$$

Example 24. *Path contraction η_Y is a homotopy $\mathrm{id}_{PY} \Rightarrow c_Y \circ t_Y$.*

The notion of homotopy between maps $f, g : X \to Y$ is different from the notion of a path in the "function space" Y^X, as can be witnessed by the following example[2] in the category of assemblies.

Example 25. *Consider an "half-line"* $\mathbf{N}$ *with the realizability realtion* $E_{\mathbf{N}}(x) = \{x, x+1\}$. *We can define two functions* $f : \mathbf{N} \to \mathbf{N}$ *and* $g : \mathbf{N} \to \mathbf{N}$

$$f(x) = 0$$

$$g(x) = x$$

realized by $\lambda x.0$ *and* $\lambda x.x$ *respectively. The functions* f *and* g *are homotopic, as witnessed by a map* $\theta : \mathbf{N} \to P\mathbf{N}$, *defined as*

$$\theta(i) = [i, x \mapsto x]$$

Then $s(\theta(i)) = 0 = f(i)$ *and* $t(\theta(i)) = i = g(i)$. *In addition,* θ *is tracked by* $\lambda x \lambda y.y$ *Thus* $\theta : f \Rightarrow g$.

However, there is no path in $P(\mathbf{N}^{\mathbf{N}})$ *that connects* f *and* g. *For suppose there is one; then it is a equivalence class with a representative* $(n, p : I_n \to \mathbf{N}^{\mathbf{N}})$ *with* $p(0) = f$ *and* $p(n) = g$. *Furthermore,* p *is tracked by some* φ. *Then a computable function* $\varphi \cdot 0$ *tracks* f, *and thus can only take values* $\{0, 1\}$, *as* $rng(f) = \{0\}$. *Since* $1 \Vdash_{\mathbf{N}} 0, 1$, *it is the case that* $\varphi \cdot 1$ *tracks both* f *and* $p(1)$. *Because it tracks* f, $\varphi \cdot 1$ *can only take values in* $\{0, 1\}$. *In particular, that means that* $rng(p(1)) \subseteq \{0, 1\}$. *By the same argument, the range of* $p(2)$ *is contained in* $\{0, 1, 2\}$. *By induction, we can prove that* $rng(p(i)) \subseteq \{0, \ldots, i\}$. *In particular,* $rng(p(n)) = rng(g) \subseteq \{0, \ldots, n\}$, *which is clearly a contradiction.*

Compositions of homotopies. Homotopies can be composed vertically. If $\theta : f \Rightarrow g$ and $\Omega : g \Rightarrow h$ are homotopies, then the vertical composition $\theta * \Omega : f \Rightarrow h$ can be visualized as depicted below.

$$\begin{tikzcd} A \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\theta"] \arrow[r, phantom, shift right=3ex, "\Downarrow\Omega"] & B \end{tikzcd} \rightarrow \begin{tikzcd} A \arrow[r, bend left=60, "f"] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, "\Downarrow\theta * \Omega"] & B \end{tikzcd}$$

[2]Example due to Jaap van Oosten, via personal communication.

It can be defined as follows. Consider

$$
\begin{tikzcd}
X \arrow[dr, dotted, "\langle\theta,\Omega\rangle"] \arrow[drr, bend left, "\Omega"] \arrow[ddr, bend right, "\theta"'] & & \\
& PY \times_Y PY \arrow[r] \arrow[d] & PY \arrow[d, "s_X"] \\
& PY \arrow[r, "t_X"] & Y
\end{tikzcd}
$$

Then $\theta * \Omega = * \circ \langle\theta, \Omega\rangle$.

We also can *whisker* a homotopy with a map; if $\theta : f \Rightarrow g$, as in the picture below

$$
\begin{tikzcd}
A \arrow[r, "h"] & X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\theta"] & Y \arrow[r, "k"] & C
\end{tikzcd}
$$

then we put

$$
\begin{aligned}
&\theta.h : f \circ h \Rightarrow g \circ h \\
&\theta.h = \theta \circ h \\
&k.\theta : k \circ f \Rightarrow k \circ g \\
&k.\theta = P(k) \circ \theta
\end{aligned}
$$

Furthermore, we can compose homotopies horizontally: if $\theta : f \Rightarrow g$ and $\Omega : h \Rightarrow k$ then $\theta \circ \Omega : h \circ f \Rightarrow k \circ g$

$$
\begin{tikzcd}
A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\theta"] & B \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\Omega"] & C & \rightarrow & A \arrow[r, bend left, "h\circ f"] \arrow[r, bend right, "k\circ g"'] \arrow[r, phantom, "\Downarrow\theta\circ\Omega"] & C
\end{tikzcd}
$$

We can obtain horizontal composition $\theta \circ \Omega$ in two ways: either as $(h.\theta) * (\Omega.g)$ or as $(\Omega.f) * (k.\theta)$. Unfortunately, there is no reasons for those operators to be equal in general; see [1, Remark 6.1.2]. For the path object category in assemblies we can obtain the following counterexample.

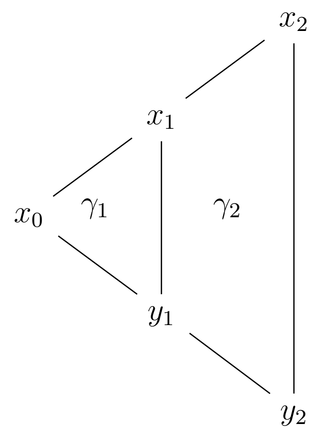

Figure 3.2: Assembly X

Consider an assembly X depicted in fig. 3.2. It consists of elements x_0, x_1, x_2, y_1, y_2, with the realizability relation $\gamma_0 \Vdash_X x_0, x_1, y_1$ and $\gamma_1 \Vdash_X x_1, y_1, y_2, x_2$, for some numbers γ_0, γ_1.

We can define two maps $\langle x_0, x_1, x_2\rangle, \langle x_0, y_1, y_2\rangle$ from the interval I_2 to X as

$$\begin{aligned} \langle x_0, x_1, x_2\rangle(0) &= x_0 & \langle x_0, y_1, y_2\rangle(0) &= x_0 \\ \langle x_0, x_1, x_2\rangle(1) &= x_1 & \langle x_0, y_1, y_2\rangle(1) &= y_1 \\ \langle x_0, x_1, x_2\rangle(2) &= x_2 & \langle x_0, y_1, y_2\rangle(2) &= y_2 \end{aligned}$$

In general, when unambiguous, we use the notation $\langle a_0, \ldots, a_n\rangle$ to denote a function $I_n \to X$ with $\langle a_0, \ldots, a_n\rangle(i) = a_i$. Both $\langle x_0, x_1, x_2\rangle$ and $\langle x_0, y_1, y_2\rangle$ are tracked by a p.r. α, which is defined as

$$\alpha(0) = \gamma_0 \quad \alpha(1) = \gamma_1 \quad \alpha(2) = \gamma_1$$

Then, we can define a homotopy $\Omega : \langle x_0, x_1, x_2\rangle \Rightarrow \langle x_0, y_1, y_2\rangle$ by

$$\begin{aligned} \Omega(0) &= c_X(x_0) \\ \Omega(1) &= [1, \langle x_1, y_1\rangle] \\ \Omega(2) &= [1, \langle x_2, y_2\rangle] \end{aligned}$$

A realizer for Ω is a p.r. φ given by

$$\varphi(0), \varphi(1) := \lambda z.\gamma_0 \qquad \varphi(2) := \lambda z.\gamma_1$$

By a straightforward calculation we get $s \circ \Omega = \langle x_0, x_1, x_2 \rangle$ and $t \circ \Omega = \langle x_0, y_1, y_2 \rangle$.

We also define maps $e_0, e_2 : \top \to I_2$ with $e_i(*) = i$ and obvious realizers. Then $\theta := * \mapsto [2, \mathrm{id}_{I_2}]$ is a homotopy between e_0 and e_2. The picture is

$$\begin{tikzcd} \top \arrow[r, bend left, "e_0"] \arrow[r, bend right, "e_2"'] \arrow[r, phantom, "\Downarrow\theta"] & I2 \arrow[r, bend left, "\langle x_0, x_1, x_2 \rangle"] \arrow[r, bend right, "\langle x_0, y_1, y_2 \rangle"'] \arrow[r, phantom, "\Downarrow\Omega"] & X \end{tikzcd}$$

Then,

$$\begin{aligned}
(\theta \circ \Omega)(*) &= (\langle x_0, x_1, x_2 \rangle.\theta)(*) * (\Omega.e_2)(*) \\
&= P(\langle x_0, x_1, x_2 \rangle)[2, \mathrm{id}_{I_2}] * \Omega(2) = [2, \langle x_0, x_1, x_2 \rangle] * [1, \langle x_2, y_2 \rangle] \\
&= [3, \langle x_0, x_1, x_2, y_2 \rangle]
\end{aligned}$$

$$\begin{aligned}
(\theta \circ_1 \Omega)(*) &= (\Omega.e_0)(*) * (\langle x_0, y_1, y_2 \rangle.\theta)(*) \\
&= \Omega(0) * P(\langle x_0, y_1, y_2 \rangle)[2, \mathrm{id}_{I_2}] = c_X(x_0) * [2, \langle x_0, y_1, y_2 \rangle] \\
&= [2, \langle x_0, y_1, y_2 \rangle]
\end{aligned}$$

Those two paths are from different equivalence classes. For, suppose $(3, \langle x_0, x_1, x_2, y_2 \rangle) \rightsquigarrow_\sigma (2, \langle x_0, y_1, y_2 \rangle)$; then $\langle x_0, x_1, x_2, y_2 \rangle(1) = x_1 = \langle x_0, y_1, y_2 \rangle(\sigma(1))$, but the latter path doesn't take value x_1 at any point.

Strong deformation retracts. Homotopies allow us to define a notion of a strong deformation retract:

Definition 26. *Given an arrow $f : X \to Y$, a* strong deformation retraction *of f is a map $k : Y \to X$, such that*

- $k \circ f = \mathrm{id}_X$
- *There is a homotopy $\theta : \mathrm{id}_Y \Rightarrow f \circ k$*

- *θ is constant on X: $\theta.f = \mathrm{id}_f$, i.e. $\theta \circ f = c_Y \circ f$*

We say that f is a strong deformation retract*, if there is a strong deformation retraction of f.*

Extended path contraction. In a nice path object category, not only we can contract a path onto its end, but we can contract a path onto a final subpath. Specifically, there is a natural transformation $L : PY \times_Y PY \to PPY$ that internally satisfies $s(L(\alpha,\beta)) = \alpha * \beta$ and $t(L(\alpha,\beta)) = \beta$. Or, equivalently, $s_{PY} \circ L = *$ and $t_{PY} \circ L = \pi_2$, where $\pi_2 : PY \times_Y PY \to PY$ is the second projection.

Here is how we construct it. First of all, we obtain a morphism $\langle \eta_Y \circ \pi_1, c_{PY} \circ \pi_2 \rangle$ from the following diagram

$$\begin{tikzcd}
PY \times_Y PY \arrow[dr, dotted, "\langle \eta_Y \circ \pi_1, c_{PY} \circ \pi_2 \rangle"] \arrow[drr, bend left, "c_{PY} \circ \pi_2"] \arrow[ddr, bend right, "\eta_Y \circ \pi_1"'] & & \\
& P(PY \times_Y PY) \arrow[r, "P(\pi_2)"] \arrow[d, "P(\pi_1)"] & PPY \arrow[d, "P(s_Y)"] \\
& PPY \arrow[r, "P(t_Y)"] & PY
\end{tikzcd}$$

In which the inner square is a pullback, since P preserves pullbacks. The outer square commutes because $P(t) \circ \eta_Y \circ \pi_1 = c_Y \circ t_Y \circ \pi_1$ by the η-law, and $P(s) \circ c_{PY} \circ \pi_2 = c_Y \circ s_Y \circ \pi_2$ by the naturality of c.

After that, we obtain the map L by the composition

$$PY \times_Y PY \xrightarrow{\langle \eta_Y \circ \pi_1, c_{PY} \circ \pi_2 \rangle} P(PY \times_Y PY) \xrightarrow{P(*)} PPY$$

First, we want to show that $sL = *$. Observe that the following diagram commutes:

$$\begin{tikzcd}
PY \times_Y PY \arrow[r, "\langle \eta_Y \circ \pi_1, c_{PY} \circ \pi_2 \rangle"] & P(PY \times_Y PY) \arrow[r, "P(*)"] \arrow[d, "s"] & PPY \arrow[d, "s"] \\
& PY \times_Y PY \arrow[r, "*"] & PY
\end{tikzcd}$$

It it thus suffices to show that $s_{PY \times_Y PY} \circ \langle \eta_Y \circ \pi_1, c_{PY} \circ \pi_2 \rangle = \mathrm{id}_{PY \times_Y PY}$. Since $PY \times_Y PY$ is a pullback, we can establish the equation by proving

$$\pi_1 \circ s_{PY \times_Y PY} \circ \langle \eta_Y \circ \pi_1, c_{PY} \circ \pi_2 \rangle = \pi_1$$
$$\pi_2 \circ s_{PY \times_Y PY} \circ \langle \eta_Y \circ \pi_1, c_{PY} \circ \pi_2 \rangle = \pi_2$$

By naturality of s, $\pi_1 \circ s_{PY \times_Y PY} = s_{PY} \circ P(\pi_1)$. Hence,

$$\pi_1 \circ s_{PY \times_Y PY} \circ \langle \eta_Y \circ \pi_1, c_{PY} \circ \pi_2 \rangle = s_{PY} \circ P(\pi_1) \circ \langle \eta_Y \circ \pi_1, c_{PY} \circ \pi_2 \rangle = s_{PY} \circ \eta_Y \circ \pi_1 = \pi_1$$

where the penultimate step holds because $P(\pi_1)$ is the first projection out of $PPY \times_{PY} PPY \simeq P(PY \times_Y PY)$. Similarly, for the second equation

$$\pi_2 \circ s_{PY \times_Y PY} \circ \langle \eta_Y \circ \pi_1, c_{PY} \circ \pi_2 \rangle = s_{PY} \circ P(\pi_2) \circ \langle \eta_Y \circ \pi_1, c_{PY} \circ \pi_2 \rangle = s_{PY} \circ c_{PY} \circ \pi_2 = \pi_2$$

This confirms that $s \circ \langle \eta_Y \circ \pi_1, c_{PY} \circ \pi_2 \rangle = \mathrm{id}$ and, therefore, $sL = *$.

Now we are going to show that $tL = \pi_2$. We use a similar strategy. First, we note that the following diagram commutes

$$\begin{tikzcd}
PY \times_Y PY \arrow[r, "\langle \eta_Y \circ \pi_1, c_{PY} \circ \pi_2 \rangle"] & P(PY \times_Y PY) \arrow[r, "P(*)"] \arrow[d, "t"] & PPY \arrow[d, "t"] \\
& PY \times_Y PY \arrow[r, "*"] & PY
\end{tikzcd}$$

If we can show that $t \circ \langle \eta_Y \circ \pi_1, c_{PY} \circ \pi_2 \rangle = \langle c_Y \circ t_Y \circ \pi_1, \pi_2 \rangle$, then we are done, because

$$*\langle c_Y \circ t_Y \circ \pi_1, \pi_2 \rangle = \pi_2$$

Because $PY \times_Y PY$ is a pullback, it suffices to verify that

$$\pi_1 \circ t \circ \langle \eta_Y \circ \pi_1, c_{PY} \circ \pi_2 \rangle = c_Y \circ t_Y \circ \pi_1$$
$$\pi_2 \circ t \circ \langle \eta_Y \circ \pi_1, c_{PY} \circ \pi_2 \rangle = \pi_2$$

Using the same tactics as in the first part, we get

$$\pi_1 \circ t \circ \langle \eta_Y \circ \pi_1, c_{PY} \circ \pi_2 \rangle = t \circ P(\pi_1) \circ \langle \eta_Y \circ \pi_1, c_{PY} \circ \pi_2 \rangle = t \circ \eta_Y \circ \pi_1 = c_Y \circ t_Y \circ \pi_1$$

and

$$\pi_2 \circ t \circ \langle \eta_Y \circ \pi_1, c_{PY} \circ \pi_2 \rangle = t \circ P(\pi_2) \circ \langle \eta_Y \circ \pi_1, c_{PY} \circ \pi_2 \rangle = t \circ c_{PY} \circ \pi_2 = \pi_2$$

Note that η is a special case of L, if we put the second component to be a constant path $c_Y(t_Y(\alpha))$.

Furthermore, we will need the following facts about L later on:

Proposition 27. *Internally, $P(s)(L(\alpha, \beta)) = \alpha$ and $P(t)(L(\alpha, \beta)) = c_Y(t_Y(\beta))$, or, equivalently*

- $P(s) \circ L = \pi_1$
- $P(t) \circ L = c_Y \circ t_Y \circ \pi_2$

Proof. Consider the following diagram,

$$\begin{tikzcd}
PY \times_Y PY \arrow[r, "\langle \eta_Y \circ \pi_1, c_{PY} \circ \pi_2 \rangle"] \arrow[dr, "c_{PY} \circ \pi_2"'] & P(PY \times_Y PY) \arrow[r, "P(*)"] \arrow[d, "P(\pi_2)"] & PPY \arrow[d, "P(t_Y)"] \\
& PYY \arrow[r, "P(t_Y)"] & PY
\end{tikzcd}$$

Where the rightmost square commutes because $P(t) \circ P(*) = P(t \circ *) = P(t \circ \pi_2)$, and the left triangle commutes because $P(\pi_2)$ is the second projection (since P preserves pullbacks).

Finally, $P(t_Y) \circ c_{PY} \circ \pi_2 = c_Y \circ t_Y \circ \pi_2$ by the naturality of c. The other equation can be verified in a similar fashion.

$\square$

Contracting a path onto the start-point. The natural transformation η allows us to contract a path onto its endpoint, i.e. $s_{PY}(\eta_Y q) = q$ and $t_{PY}(\eta_Y q) = c_Y(t_Y q)$. We may also wish to contract a path onto its starting point, i.e. have a homotopy $\nu_Y : \mathrm{id}_{PY} \Rightarrow c_Y \circ s_Y$. We obtain it through as a composition $\nu_Y := P(\tilde{\cdot}) \circ \eta_Y \circ \tilde{\cdot} : PY \to PPY$. Indeed, the diagram

$$\begin{tikzcd}
& & PY \arrow[r, "\tilde{\cdot}"] & PY \\
PY \arrow[r, "\tilde{\cdot}"] & PY \arrow[ur, bend left, "\mathrm{id}_{PY}"] \arrow[r, "\eta_Y"] \arrow[dr, bend right, "c_Y \circ t_Y"'] & PPY \arrow[u, "s_Y"] \arrow[r, "P(\tilde{\cdot})"] \arrow[d, "t_Y"] & PPY \arrow[u, "s_{PY}"'] \arrow[d, "t_{PY}"] \\
& & PY \arrow[r, "\tilde{\cdot}"] & PY
\end{tikzcd}$$

commutes. Hence, $s_{PY} \circ \nu_Y = \tilde{\cdot} \circ \mathrm{id}_{PY} \circ \tilde{\cdot} = \mathrm{id}_{PY}$ and $t_{PY} \circ \nu_Y = \tilde{\cdot} \circ c_Y \circ t_Y \circ \tilde{\cdot} = c_Y \circ s_Y$. In a similar way we can prove the other counterparts of the axioms in definition 18. For instance, we can derive the following equations:

$$P(s_Y) \circ \nu_Y = c_Y \circ s_Y \quad P(t_Y) \circ \nu_Y = \tilde{\cdot}$$
$$\nu_Y \circ c_Y = c_{PY} \circ c_Y$$

3.3 Factorisation system

In a nice path object category, every map can be factored as a strong deformation retract followed by a Hurewicz fibration (notions which we will define shortly, definitions 26 and 28). Furthermore, such factorisation is functorial.

Let $f : X \to Y$ be a morphism. We define the factorisation by taking the pullback

$$\begin{tikzcd} Ef \arrow[r, "q_f"] \arrow[d, "p_f"] & PY \arrow[d, "t_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{3.1}$$

Then, since $t_Y \circ c_Y \circ f = f$, we obtain a map $\lambda_f = \langle \mathrm{id}_X, c_Y \circ f \rangle : X \to Ef$ from the universal property of the pullback

$$\begin{tikzcd} X \arrow[dr, dotted, "\lambda_f"] \arrow[drr, bend left, "c_Y \circ f"] \arrow[ddr, bend right, "\mathrm{id}_X"'] & & \\ & Ef \arrow[r, "q_f"] \arrow[d, "p_f"] & PY \arrow[d, "t_Y"] \\ & X \arrow[r, "f"] & Y \end{tikzcd}$$

Furthermore, we put $\rho_f := s_Y \circ q_f$. This data gives us a factorisation

$$X \xrightarrow{\lambda_f} Ef \xrightarrow{\rho_f} Y$$

of $f : X \to Y$, because $\rho_f \circ \lambda_f = s_Y \circ q_f \circ \lambda_f = s_Y \circ c_Y \circ f = f$.

To see that this construction is functorial, suppose we have a commutative square

$$
\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, "\lambda_f"] & W \arrow[d, "\lambda_g"] \\
Ef \arrow[d, "\rho_f"] & Eg \arrow[d, "\rho_g"] \\
Y \arrow[r, "k"] & Z
\end{tikzcd}
$$

Then we are to find a filler $E(h,k) : Ef \to Eg$. We construct it from the pullback square

$$
\begin{tikzcd}
Ef \arrow[dr, dashed, "P(h,k)"] \arrow[drr, bend left, "P(k)\circ q_f"] \arrow[ddr, bend right, "h\circ p_f"'] & & \\
& Eg \arrow[r, "q_g"] \arrow[d, "p_g"] & PZ \arrow[d, "t_Z"] \\
& W \arrow[r, "g"] & Z
\end{tikzcd}
$$

It is possible to see that this indeed is a filler by using the universal property of a pullback; see [2, Lemma 6.6] for details.

Next we give a definition of a Hurewicz fibration.

Definition 28 (Hurewicz fibration [7]). *Given a map f, we obtain a map v_f from the diagram*

$$
\begin{tikzcd}
PX \arrow[dr, dashed, "v_f"] \arrow[drr, bend left, "P(f)"] \arrow[ddr, bend right, "t_X"'] & & \\
& Ef \arrow[r, "q_f"] \arrow[d, "p_f"] & PY \arrow[d, "t_Y"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}
$$

The morphism f is a Hurewicz fibration *if v_f has a section. Such a section is called a* Hurewicz connection.

Hurewicz fibrations can be characterized by the homotopy lifting property, which, when rewritten in the language of path object categories, states the following:

Definition 29. *A map $f : X \to Y$ has a homotopy lifting property if for any map $g : Z \to X$ and a homotopy $\theta : Z \to PY$, such that $t\theta = fg$,*

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "\theta"] & X \arrow[d, "f"] \\ PY \arrow[r, "t"] & Y \end{tikzcd}$$

there is a lift $\bar{\theta} : Z \to PX$ such that $t\bar{\theta} = g$ and $f.\bar{\theta} = P(f) \circ \bar{\theta} = \theta$.

Just like in standard case of topological spaces, the covering homotopy condition is both sufficient and necessary for a map to be a Hurewicz fibration

Lemma 30 (Covering homotopy condition)**.** *A map $f : X \to Y$ is a Hurewicz fibration/has a Hurewicz connection iff it has the covering homotopy property.*

Proof. ($\Rightarrow$). Suppose λ is a section of v_f. The desired lifting is then obtained as $\bar{\theta} = \lambda \circ \langle g, \theta \rangle$, where $\langle g, \theta \rangle$ is obtained from the pullback square

$$\begin{tikzcd} Z \arrow[dr, dashed, "\langle g,\theta\rangle"] \arrow[drr, bend left, "\theta"] \arrow[ddr, bend right, "g"'] & & \\ & Ef \arrow[r, "q_f"] \arrow[d, "p_f"] & PY \arrow[d, "t_Y"] \\ & X \arrow[r, "f"] & Y \end{tikzcd}$$

Then $t_X\bar{\theta} = t_X \circ \lambda \circ \langle g, \theta \rangle = p_f \circ \langle g, \theta \rangle = g$ and $p.\bar{\theta} = P(p) \circ \lambda \circ \langle g, \theta \rangle = q_f \circ \langle g, \theta \rangle = \theta$.

($\Leftarrow$). Put $\theta := q_f : Ef \to PY$ and $g := p_f : Ef \to X$; clearly, $t\theta = f \circ g$. Then, the lifted homotopy $\bar{\theta} : Ef \to PX$ satisfies $t_X\bar{\theta} = p_f$ and $P(p)\circ\bar{\theta} = q_f$. Thus, the following diagram commutes:

$$\begin{tikzcd}
Ef \arrow[dr, "\bar{\theta}"] \arrow[ddrrr, bend left, "q_f"] \arrow[dddrr, bend right, "p_f"'] & & & \\
& PX \arrow[dr, dashed, "v_f"] \arrow[drr, bend left, "P(p)"] \arrow[ddr, bend right, "t_X"'] & & \\
& & Ef \arrow[r, "q_f"] \arrow[d, "p_f"] & PY \arrow[d, "t_Y"] \\
& & X \arrow[r, "f"] & Y
\end{tikzcd}$$

Since Ef is a pullback, we can conclude that $v_f \circ \bar{\theta} = \mathrm{id}_{Ef}$. □

Finally, we are going to show that the functorial factorisation, defined in the beginning of this section, factors each map as a strong deformation retract followed by a Hurewicz fibration.

Proposition 31. *For any map f, λ_f is a strong deformation retract and ρ_f is a Hurewicz fibration.*

Proof. Note that p_f is a retraction of λ_f; to show that it is a strong deformation retraction we must find a homotopy θ, that satisfies the conditions of definition 26. For this we take the image of eq. (3.1) under P; since P preserves pullbacks, the resulting diagram is a pullback as well. Thus, to obtain a map $Ef \to P(Ef)$ is to obtain a pair of maps $Ef \to PX$ and $Ef \to PPY$.

Note, that the following diagram commutes

$$\begin{tikzcd}
Ef \arrow[r, "q_f"] \arrow[d, "p_f"] & PY \arrow[r, "\eta_Y"] \arrow[d, "t_Y"] & PPY \arrow[d, "P(t)"] \\
X \arrow[r, "f"] \arrow[dr, "c_X"'] & Y \arrow[r, "c_Y"] & PY \\
& PX \arrow[ur, "P(f)"'] &
\end{tikzcd}$$

in which the upper left square is the pullback eq. (3.1), the upper right square is a law for η, and the lower triangle expresses the naturality of c.

From that diagram we obtain $\theta : Ef \to P(Ef)$ as

$$
\begin{tikzcd}
Ef \arrow[dr, dotted, "\theta"] \arrow[drr, bend left, "\eta_Y \circ q_f"] \arrow[ddr, bend right, "c_X \circ p_f"'] & & \\
& P(Ef) \arrow[r, "P(q_f)"] \arrow[d, "P(p_f)"] & PY \arrow[d, "P(t_Y)"] \\
& X \arrow[r, "P(f)"] & Y
\end{tikzcd}
$$

We have to show that $s\theta = \mathrm{id}_{Ef}$ and $t\theta = \lambda_f \circ p_f$; for this it is sufficient to verify those identities up to the composition with p_f and q_f, since Ef is a pullback, i.e.

$$
\begin{cases}
p_f \circ s_{Ef} \circ \theta = p_f \\
q_f \circ s_{Ef} \circ \theta = q_f
\end{cases}
$$

and

$$
\begin{cases}
p_f \circ t_{Ef} \circ \theta = p_f \circ \lambda_f \circ p_f = p_f \\
q_f \circ t_{Ef} \circ \theta = q_f \circ \lambda_f \circ p_f = c_Y \circ f \circ p_f = c_Y \circ t_Y \circ q_f
\end{cases}
$$

For the first set of equations, note that $p_f \circ s_{Ef} = s_X \circ P(p_f)$ by naturality, hence

$$
p_f \circ s_{Ef} \circ \theta = s_X \circ P(p_f) \circ \theta = s_X \circ c_X \circ p_f = p_f
$$

and similarly for q_f. For the second set of equations, we employ the same strategy. By naturality, $p_f \circ t_{Ef} = t_X \circ P(p_f)$ and $q_f \circ t_{Ef} = t_X \circ P(q_f)$, hence

$$
p_f \circ t_{Ef} \circ \theta = t_X \circ c_X \circ p_f = p_f
$$

and

$$
q_f \circ t_{Ef} \circ \theta = t_Y \circ \eta_Y \circ q_f = c_Y \circ t_Y \circ q_f
$$

Finally, we have to check that θ is constant at λ_f; that is $\theta \circ \lambda_f = c_{Ef} \circ \lambda_f$. Once again, it suffices to note that

$$
\begin{cases}
P(p_f) \circ \theta \circ \lambda_f = c_X \circ p_f \circ \lambda_f = P(p_f) \circ c_{Ef} \circ \lambda_f = c_X \circ p_f \circ \lambda_f \\
P(q_f) \circ \theta \circ \lambda_f = \eta_Y \circ q_f \circ \lambda_f = \eta_Y \circ c_Y \circ f = P(q_f) \circ c_{Ef} \circ \lambda_f = c_{PY} \circ q_f \circ \lambda_f = c_{PY} \circ c_Y \circ f
\end{cases}
$$

where the last equation holds due to an η law.

Now we shall show that ρ_f is a Hurewicz fibration. Since E_{ρ_f} is a pullback, to find a section k of v_{ρ_f} is to find a map k making the following diagram commute:

$$\begin{tikzcd}
E_{\rho_f} \arrow[dr, dashed, "k"] \arrow[drrr, bend left, "q_{\rho_f}"] \arrow[ddrr, bend right, "p_{\rho_f}"'] & & & \\
& PEf \arrow[dr, dashed, "v_{\rho_f}"] \arrow[drr, bend left, "P(\rho_f)"] \arrow[ddr, bend right, "t_X"'] & & \\
& & E_{\rho_f} \arrow[r, "q_{\rho_f}"] \arrow[d, "p_{\rho_f}"] & PY \arrow[d, "t_Y"] \\
& & Ef \arrow[r, "\rho_f"] & Y
\end{tikzcd}$$

i.e. to find a map k satisfying

$$\begin{cases} t \circ k = p_{\rho_f} \\ P(\rho_f) \circ k = q_{\rho_f} \end{cases}$$

We construct this map in two stages. First of all, we define a map $m : E_{\rho_f} \to PY \times_Y PY$ from the following diagram

$$\begin{tikzcd}
E_{\rho_f} \arrow[dr, dotted, "m"] \arrow[drr, bend left, "q_f \circ p_{\rho_f}"] \arrow[ddr, bend right, "q_{\rho_f}"'] & & \\
& PY \times_Y PY \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & PY \arrow[d, "s_Y"] \\
& PY \arrow[r, "t_Y"] & Y
\end{tikzcd} \tag{3.2}$$

Then $L \circ m$ is a homotopy $* \circ m \Rightarrow q_f \circ p_{\rho_f}$. Internally, E_{ρ_f} is an object of "triples" (x, α, β), such that $t\beta = s\alpha$ and $t\alpha = f(x)$. Then m picks out (β, α) and L contracts $\beta * \alpha$ onto α.

Secondly, we obtain k from the pullback $P(Ef)$ as in the diagram below.

$$\begin{tikzcd}
E_{\rho_f} \arrow[r, "m"] \arrow[d, "p_{\rho_f}"'] \arrow[dr, dotted, "k"] & PY \times_Y PY \arrow[dr, "L"] & \\
Ef \arrow[d, "p_f"'] & PEf \arrow[r, "P(q_f)"] \arrow[d, "P(p_f)"] & PPY \arrow[d, "P(t)"] \\
X \arrow[r, "c_X"] & PX \arrow[r, "P(f)"] & PY
\end{tikzcd}$$

The outer part of the diagram commutes because

$$\begin{aligned}
P(t) \circ L \circ m &= c_Y \circ t_Y \circ \pi_2 \circ m && \text{proposition 27} \\
&= c_Y \circ t_Y \circ q_f \circ p_{\rho_f} && \text{(eq. (3.2))} \\
&= c_Y \circ f \circ p_f \circ p_{\rho_f} && \text{(eq. (3.1))} \\
&= P(f) \circ c_X \circ p_f \circ p_{\rho_f} && \text{naturality of } c
\end{aligned}$$

Now we are going to verify that

$$\begin{cases} t \circ k = p_{\rho_f} \\ P(\rho_f) \circ k = q_{\rho_f} \end{cases}$$

- Equation $t \circ k = p_{\rho_f}$: Because Ef is a pullback, it is sufficient to show that

  $$\begin{cases} p_f \circ t \circ k = p_f \circ p_{\rho_f} \\ q_f \circ t \circ k = q_f \circ p_{\rho_f} \end{cases}$$

  For this, note that the following diagrams commute

  $$\begin{tikzcd}
  E_{\rho_f} \arrow[r, "k"] \arrow[d, "p_f \circ p_{\rho_f}"'] & PEf \arrow[r, "t"] \arrow[d, "P(p_f)"] & Ef \arrow[d, "p_f"] \\
  X \arrow[r, "c_X"] & PX \arrow[r, "t_X"] & X
  \end{tikzcd}
  \qquad
  \begin{tikzcd}
  E_{\rho_f} \arrow[r, "k"] \arrow[d, "m"'] & PEf \arrow[r, "t"] \arrow[d, "P(q_f)"] & Ef \arrow[d, "q_f"] \\
  PY \times_Y PY \arrow[r, "L"] & PPY \arrow[r, "t_{PY}"] & PY
  \end{tikzcd}$$

  Thus,

  $$\begin{aligned}
  p_f \circ t \circ k &= t_X \circ c_X \circ p_f \circ p_{\rho_f} = p_f \circ p_{\rho_f} \\
  q_f \circ t \circ k &= t_{PY} \circ L \circ m = \pi_2 \circ m \circ q_f \circ p_{\rho_f}
  \end{aligned}$$

- Equation $P(\rho_f) \circ k = q_{\rho_f}$: We have

$$P(\rho_f) \circ k = P(s_Y) \circ P(q_f) \circ k = P(s) \circ L \circ m = \pi_1 \circ = q_{\rho_f}$$

$\square$

3.4 Discrete reflection

We can think of some assemblies (and of some objects in the effective topos) as being *discrete*, meaning that they do not have any non-trivial paths. In *Asm* such objects are modest sets.

Definition 32. *An assembly* (X, E) *is a* modest set *if for each set* $E(x)$, $x \in X$ *is a singleton.*

Theorem 33. *Modest sets are* discrete *in the sense that for all* $n \geq 1$, *there are no non-constant paths* $p : I_n \to M$.

Proof. Suppose p is such a path and $p(i) \neq p(j)$ for some $i < j$. Let e be the realizer/tracker for p. Then there is a number $i' \in [i, j)$, s.t. $p(i') \neq p(i'+1)$. Since $i'+1 \Vdash i', i'+1$, it must be the case that $e(i'+1) \Vdash p(i'), p(i'+1)$. That means that $E(p(i')) \cap E(p(i'+1))$ is non-empty, which is impossible, since M is a modest set. $\square$

Theorem 34. *The converse holds as well: if* X *is discrete, then* X *is modest.*

Proof. Let $x, y \in X$ and $x \neq y$. Define a set-level function $p : I_1 \to X$ by $p(0) = x$ and $p(1) = y$. It is easy to see that p is realized iff there is some $c \in E_X(x) \cap E_X(y)$ – the realizer would be a p.r. function defined by $\lambda x.c$. $\square$

The inclusion functor $Mod \hookrightarrow Asm$ has a left adjoint, giving rise to a *discrete reflection.*

Suppose X is an assembly. We define its discrete reflection X_d as a coequalizer of the following diagram:

$$PX \underset{t}{\overset{s}{\rightrightarrows}} X$$

Concretely, X_d is the quotient of X by the relation $\sim_p$, such that $sp \sim_p tp$ for all $p \in PX$. In other words, $x \sim_p y$ if there is a path $p' : I_m \to X$, with

$p'(0) = x$ and $p'(m) = y$. Intuitively, a class $[x] \in X_d$ contains all points of X that are path-connected to x. The realizability relation on X_d is defined as

$$a \Vdash_{X_d} [x] \iff a \Vdash_X y \text{ for some } y \in [x]$$

The map $[-] : X \to X_d$ is realized by $\lambda x.x$.

Given a morphism $f : X \to Y$ (tracked by $\underline{f}$), we obtain a morphism $f_d : X_d \to Y_d$ defined as

$$f_d([x]) = [f(x)]$$

and realized by $\underline{f}$. To establish the reflection we have to verify two things: that X_d is indeed discrete, and that the arrow $[-]$ is universal.

For the first part, suppose that $[x], [y] \in X_d$, $[x] \neq [y]$. Then, by definition, $x \notin [y]$, and there is no path between x and y in X; in particular, there is no path $p' : I_1 \to X$, such that $sp' = x$ and $tp' = y$. Thus, essentially by the same argument as in theorem 34, X_d is modest.

The second part amounts to filling in a dotted morphism g in the following diagram, where Y is modest/discrete:

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, "{[-]}"] \\ Y & X_d \arrow[l, dotted, "g"] \end{tikzcd}$$

We simply put $g([x]) = f(x)$. This is well-defined, for suppose $y \in [x]$, i.e. there is a path $p' : I_m \to X$ connecting x and y. Then, by the same argument as in theorem 34, there is a $c \in E_X(x) \cap E_X(y)$. Given a realizer $\underline{f}$ for f, we have $\underline{f} \cdot c \Vdash_Y f(x), f(y)$. Because Y is modest, $f(x) = f(y)$. Finally, if $a \Vdash_{X_d} [x]$, then $a \Vdash_X y$ for some $y \in [x]$, and $\underline{f} \cdot a \Vdash_Y f(y) = g([x])$.

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