FROBENIUS PROPERTY OF A WEAK FACTORISATION SYSTEM

Abstract. In this note I would like to show that if a locally Cartesian closed category with the type structure induced by a weak factorisation system supports Π-types, then the factorisation system has Frobenius property.

This was recently communicated to me by Benno van den Berg, who mentioned that this idea is possibly folklore. In any way, I didn't manage to find the material outlined below published anywhere, so I decided to typeset this and put it up online. So, none of this is original.

A weak factorisation system (alternatively, a cloven factorisation system, and algebraic weak factorisation system, etc) is said to have a *Frobenius property* [2, 3.3.3(iv)], if cofibrations are stable under the pullbacks along fibrations. That is, given a pullback

$$\begin{array}{ccc} f^*(A) & \longrightarrow & A \\ {\scriptstyle \bar{i}}\big\downarrow & & \big\downarrow{\scriptstyle i} \\ \Delta & \xrightarrow{\ f\ } & \Gamma \end{array}$$

if i is a cofibration and f is a fibration, then $\bar{i}$ is also a cofibration.

This property has a connection the axioms for identity types in Martin-Löf type theory, as interpreted in categories with weak factorisation systems as in [1]. In particular, this property is useful for models of type theory without Π-types. To see this, consider the usual rule for Id-elimination

$$\frac{x : A, y : A, u : \mathrm{Id}_A(x, y) \vdash P(x, y, u) \text{ type} \qquad x : A \vdash d(x) : P(x, x, r(x))}{x : A, y : A, p : \mathrm{Id}_A(x, y) \vdash J(d, x, y, p) : P(x, y, p)}$$

Under this formulation the type P can only depend on x, y and u. We want to allow P to depend on other arbitrary types and terms as well. Thus, we can reformulate the elimination rule as show in fig. 1

$$\frac{x : A, y : A, u : \mathrm{Id}_A(x, y), \Delta \vdash P(x, y, u) \text{ type} \qquad x : A, \Delta \vdash d(x) : P(x, x, r(x))}{x : A, y : A, p : \mathrm{Id}_A(x, y), \Delta \vdash J(d, x, y, p) : P(x, y, p)}$$

Figure 1. Modified Id elimination rule

This rule is valid in the model if it supports Frobenius property. The term J arises as a solution to the problem of lifting a cofibration $r : X \to X.X.\mathrm{Id}(X)$ against a fibration $P : X.X.\mathrm{Id}(X).P \to X.X.\mathrm{Id}(X)$

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A short note by Dan Frumin.

$$\begin{array}{ccc} X & \xrightarrow{[\mathrm{id},\mathrm{id},r,d]} & X.X.\operatorname{Id}(X).P \\ {\scriptstyle r}\downarrow & \overset{J}{\nearrow} & \downarrow{\scriptstyle P} \\ X.X.\operatorname{Id}(X) & = & X.X.\operatorname{Id}(X) \end{array}$$

If the type P depends on Δ, then we have a fibration $X.X.\operatorname{Id}(X).\Delta.P \to X.X.\operatorname{Id}(X).\Delta$ and if we want to lift r against it we have to weaken the context of r, by pulling it back along the fibration/weakening map $\Delta : X.X.\operatorname{Id}(X).\Delta \to X.X.\operatorname{Id}(X)$.

$$\begin{array}{ccc} X.\Delta & \longrightarrow & X \\ {\scriptstyle \bar{r}}\downarrow & & \downarrow{\scriptstyle r} \\ X.X.\operatorname{Id}(X).\Delta & \xrightarrow{\Delta} & X.X.\operatorname{Id}(X) \end{array}$$

By the Frobenius property, $\bar{r}$ is still a cofibration, so we can lift it against P.

$$\begin{array}{ccc} X.\Delta & \longrightarrow & X.X.\operatorname{Id}(X).\Delta.P \\ {\scriptstyle \bar{r}}\downarrow & \overset{J}{\nearrow} & \downarrow{\scriptstyle P} \\ X.X.\operatorname{Id}(X).\Delta & = & X.X.\operatorname{Id}(X).\Delta \end{array}$$

In the presence of Π-types, the rule fig. 1 is derivable. For suppose $P(x, y, u, \delta)$ is a type in a context $x : A, y : A, u : \operatorname{Id}(x, y), \delta : \Delta$. Then we can form a type $\Pi_{\delta:\Delta} P(x, y, u, \delta)$ in a stronger context $x : A, y : A, u : \operatorname{Id}(x, y)$, with which we can apply the standard Id elimination rule. In fact, if your model is a locally Cartesian closed category, and it supports Π-types – that is, fibrations are closed under Π_f, where f is a fibration – then the Frobenius property is derivable.

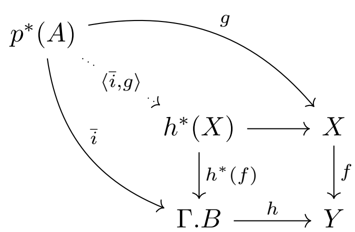
Suppose we have a pullback

$$\begin{array}{ccc} p^*(A) & \longrightarrow & A \\ \downarrow{\scriptstyle \bar{i}} & & \downarrow{\scriptstyle i} \\ \Gamma.B & \xrightarrow{p} & \Gamma \end{array}$$

where i is a cofibration and p is a fibration. To show that $\bar{i}$ is a cofibration as well it is sufficient to provide a solution to an arbitrary lifting problem

$$\begin{array}{ccc} p^*(A) & \xrightarrow{g} & X \\ \downarrow{\scriptstyle \bar{i}} & & \downarrow{\scriptstyle f} \\ \Gamma.B & \xrightarrow{h} & Y \end{array} \tag{1}$$

with f being a fibration. First of all, we pull back f along h and observe that there is a morphism $\langle \bar{i}, g\rangle : p^*(A) \to h^*(X)$ making the obvious diagrams commute.



As a reader can verify, we reduced the problem of finding a solution to the lifting problem 1, to finding the a solution to the following lifting problem

(2)
$$\begin{array}{ccc} p^*(A) & \xrightarrow{\langle \bar{i}, g\rangle} & h^*(X) \\ \big\downarrow{\scriptstyle \bar{i}} & & \big\downarrow{\scriptstyle h^*(f)} \\ \Gamma.B & = \!= \!= & \Gamma.B \end{array}$$

Now we can use the $p^* \dashv \Pi_p$ adjunction

$$\frac{\mathcal{C}/\Gamma.B\colon\ \bar{i} \to h^*(f)}{\mathcal{C}/\Gamma\colon\ i \to \Pi_p(h^*(f))}$$

to obtain another commutative square

$$\begin{array}{ccc} A & \xrightarrow{\overline{\langle \bar{i}, g\rangle}} & \Pi_p h^*(X) \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle \Pi_p h^*(f)} \\ \Gamma & = \!= \!= & \Gamma \end{array}$$

Then, since Π_p preserves fibrations, and i is a cofibration, we have a lift $j : \Gamma \to \Pi_p(h^*(X))$ making the diagram above commute. The arrow j can also be seen as a map $j : \mathrm{id}_\Gamma \to \Pi_p(h^*(f))$ in $\mathcal{C}/\Gamma$. Using the adjunction we obtain a map $\bar{j} : \mathrm{id}_{\Gamma.B} \to h^*(f)$ in $\mathcal{C}/\Gamma.B$. Then $\overline{\langle \bar{i}, g\rangle} = \overline{j \circ i} = \bar{j} \circ \bar{i}$ by the naturality of the adjunction. Hence $\bar{j}$ is the solution to the lifting problem 2.

It was shown in [3, Proposition 14] that a classifying category $C(\mathbb{T})$ for a type theory $\mathbb{T}$ with identity types admits a weak factorisation system with Frobenius property; the authors explicitly use a modified Id-types rules, because they are working in a system without Π-types (see [3, Remark 3]). Frobenius property was also used in [2] and [4]. I would like to know the history of the name and a relation, if there is one, to the Frobenius reciprocity.

References

[1] Steve Awodey and Michael A. Warren. "Homotopy theoretic models of identity types." In: *Math. Proc. Cambridge Philos. Soc.* 146.1 (2009), pp. 45–55.

[2] Benno van den Berg and Richard Garner. "Topological and Simplicial Models of Identity Types." In: *ACM Trans. Comput. Logic* 13.1 (Jan. 2012), 3:1–3:44.

[3] Nicola Gambino and Richard Garner. "The identity type weak factorisation system." In: *Theoret. Comput. Sci.* 409.1 (2008), pp. 94–109.

[4] B. van den Berg and R. Garner. "Types are weak omega-groupoids." In: *ArXiv e-prints* (Dec. 2008).